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Yuliyan Mitkov Ulrich Schüwer

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On the Relationship between Borrower and Bank risk*

Yuliyan Mitkov
University of Bonn
ymitkov@uni-bonn.de

Ulrich Schüwer
Goethe University Frankfurt
schuewer@finance.uni-frankfurt.de

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Abstract

We use tools from survival analysis to study the equilibrium probability of bank failure in a model with imperfect correlation in loan defaults where a systematic risk factor and idiosyncratic frailty factors govern borrower credit worth. We derive several surprising results: in equilibrium, a bank can be more likely to fail with less risky than with more risky borrowers. In addition, the equilibrium relationship between borrower and bank risk can be fundamentally altered by a greater dispersion of the frailty factors, similar to how mixing items of different durability can fundamentally change the overall aging pattern.

Keywords: Correlated defaults, borrower heterogeneity, bank failure, survival analysis

JEL Codes: G21, G28, E43

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1 Introduction

It is conventional wisdom that safer borrowers are associated with safer banks. If this association is not there or even reversed, the real cause of greater bank risk can be traced to lax regulation and banking supervision, changing market environment, or generous government guarantees.[1] Far from causing greater bank risk, safer borrowers act as a buffer and partially offset these other factors, similar to how having more bank equity makes the bank safer by acting as a buffer absorbing loan losses.

We argue that less risky borrowers can induce a higher probability of bank failure. This *unconventional outcome* happens because lower credit risk generates two effects that operate simultaneously and in opposing directions whenever loan defaults are imperfectly correlated: fewer non-performing loans but also lower income from each performing loan since the loan interest rate is lower. The second effect is present as long as the loan interest rates and credit quality are inversely related in equilibrium, which is the case whenever the credit market is at least partially competitive. However, the exact nature of competition is immaterial to the point we wish to make, which sets our paper apart from the extensive literature on the relationship between competition and bank failure risk (Keeley, 1990; Boyd and De Nicolo, 2005; Martinez-Miera and Repullo, 2010). Similarly, the channel we investigate arises independently of miss-priced government guarantees, which are well-known for distorting bank choices along multiple dimensions (see Kareken and Wallace (1978) and the subsequent literature).

While we are not the first to highlight the presence of these two effects (see, e.g., Repullo and Suarez (2004)), we show that their interaction with the probability of bank failure has not been fully explored. Specifically, the canonical approach to modeling correlated defaults is based on the conditional default probability framework, where all borrowers are exposed to one or more systematic risk factors (see, e.g., Vasicek (2002); Gordy (2000); Frey and McNeil (2003)). However, the existing literature on the *equilibrium* relationship between borrower and bank risk building on this framework restricts attention to specific functional forms and usually abstracts from heterogeneous borrower exposure to the systematic risk factors (e.g., Gornall and Strebulaev (2018); Nagel and Purnanandam (2020); Mendicino et al. (2019)). It should be noted that, by heterogeneous exposure, we mean borrower-specific factor loadings in a credit risk model.[2] While few would doubt that such borrower heterogeneity is important

[1]Safer borrowers tend to be larger, with more cash, less volatile cash flows, better collateral, and easier for outside investors to evaluate. However, we do not need to take a stance in this paper on what exactly makes a borrower safer.

[2]That is, we are interested in situations where the credit quality of the i-th borrower is $X_i = \sum_{i=1}^{J} \theta_{i,j} Z_j + \sigma_i \epsilon_i$ where $Z_1 ... \ Z_J$ are systematic risk factors, the coefficients $\theta_{i,j} ... \ \theta_{i,J}$ are borrower-specific factor loadings

for understanding bank risk, one major obstacle is methodological since the analysis quickly becomes intractable.

We show how one can apply survival analysis, especially mixture hazard rates, to circumvent some of these issues and systematically investigate how borrower heterogeneity shapes bank risk. Then, to illustrate the perils of neglecting borrower heterogeneity in models featuring correlated defaults, we will demonstrate how the equilibrium relationship between borrower and bank risk can be fundamentally altered when the borrowers are heterogeneously exposed to the systematic risk factor through the presence of *frailty* factors (similar to how some people are more resistant to certain types of viruses than others). In particular, we compare the equilibrium probability of bank failure in two economies, A and B, under two scenarios. *Scenario 1*: the frailty factor for each borrower in each economy is fixed at the same level. *Scenario 2*: The frailty factor varies among the borrowers according to some distribution, which is the same in both economies. For concreteness, suppose each borrower under *Scenario 2* can have either low or high frailty with equal probability (and frailty realizations are i.i.d. across all borrowers in each economy). We then derive conditions for the emergence of the following outcome. Under *Scenario 1*, the bank in Economy A is *more* likely to fail than the bank in Economy B for *each* possible value of the frailty factor. However, under *Scenario 2*, this reverses, and now the bank in Economy A is *less* likely to fail than in Economy B.

In other words, mixing borrowers with different frailties can fundamentally alter how credit worth is associated with the underlying systematic risk factor. Moreover, this change is not innocuous, as it can have a significant effect on the risk of bank failure. We refer to such outcomes as bank *risk reversals* since the situation is similar to how mixing items of different durability can fundamentally change the observed pattern of aging in survival analysis (Barlow and Proschan, 1975).[3] To our knowledge, we are the first to point out that mixing borrowers with different frailties can have such perverse effects on a bank's risk of failure.

Methodologically, we will treat the systematic risk factor as 'time' and the frailty factor as the mixing variable. This allows us to bring tools developed in the context of survival analysis used to study mixture distributions.[4] Mixtures arise whenever there is some unobservable

(i.e., borrower-specific frailty factors) and ϵ_i is a borrower-specific shock. Such heterogeneity is built into widely used credit risk models, such as CreditMetrics and KMV (Frey and McNeil, 2003). Loan defaults will be imperfectly correlated whenever $\sigma_i > 0$ whereas homogeneous exposure corresponds to $\theta_{i,j} = \theta_j$ for all i.

[3]It is also related to the Simpson paradox whereby a given association in each sub-population is reversed once all sub-populations are mixed (Simpson, 1951; Lindley and Novick, 1981).

[4]The hazard rate approach has a long history in credit risk modeling. However, the focus is on the credit risk of individual loans rather than correlated defaults (see, e.g., Duffie and Singleton (2003) and the references therein).

heterogeneity (i.e., frailty) in the objects of study. One can hardly find situations in the real world where such heterogeneity can safely be assumed away. This is especially true in the study of credit risk due to borrower-specific frailty factors, in addition to systematic risk factors, that are difficult or impossible to anticipate and contract upon. In some cases, banks may not even be legally allowed to discriminate based on certain borrower characteristics.

It is well-known that a mixture of decreasing hazard rate distributions also has a decreasing hazard rate (Barlow and Proschan, 1975). At the same time, a mixture of increasing hazard rate distributions does not necessarily have an increasing hazard rate and may have a strictly decreasing hazard rate and, in general, highly non-monotone hazard rates (Block et al., 2003). This can occur even when the component distributions have rapidly (i.e., exponentially) increasing hazard rates, thus defying intuition (Gurland and Sethuraman, 1995). In other words, the hazard rate order is not closed under mixtures, unlike, for example, the usual stochastic order (i.e., first-order stochastic dominance). The famous borderline case in Proschan (1963) is that any mixture of constant hazard rate distributions, namely mixtures of exponentials, displays a strictly decreasing hazard rate.

The counter-intuitive properties of mixture hazard rates can be traced to the changing composition of the mixture and, in particular, to what has been suggestively termed the *weak-die-first-effect* (Vaupel et al., 1979; Finkelstein and Esaulova, 2006). The mixture hazard rate equals the average hazard rate among the surviving items. As time passes, the lower-durability items are more likely to die out, whereas the higher-durability items are more likely to survive. The mixture hazard rate can thus decrease even though each item has an increasing hazard rate (and thus becomes more likely to die as time passes) because those who survive items tend to have higher durability than those who die. In our case, the weak-die-first-effect would manifest as the changing composition of solvent borrowers as a function of the systematic risk factor.

Our paper relates to the growing literature on the equilibrium relationship between credit risk and bank failure probability. The main feature of loan portfolios is an imperfect correlation in defaults, which can be captured by assuming that the borrowers are exposed to one or more systematic risk factors (Schönbucher, 2001; Gordy, 2003; Frey and McNeil, 2003). Several recent papers building on risk factor models have shown that models of bank risk that do not take into account this special feature of bank assets will tend to generate misleading implications for bank failure risk (e.g., Gornall and Strebulaev (2018); Nagel and Purnanandam (2020); Mendicino et al. (2019), but all these papers assume that the borrowers are homogeneously exposed to the systematic risk factor. We add to this literature by showing that the equilibrium relationship between borrower and bank risk is significantly more complicated than previously thought, especially when, more realistically, the borrowers are heterogeneity exposed to the systematic risk factor.

2 Environment and hazard rate preliminaries

This section describes the environment (i.e., the borrowers and the banks) followed by preliminaries on hazard rate functions. The emphasis is on making minimal assumptions, only those necessary to streamline the analysis, so that the results can be applied to various credit risk models. There are two dates (0 and 1) and N homogeneous borrowers on date 0. Each borrower gets a loan of \$1 on date 0 and must repay $1 + r$ on date 1, where the loan interest rate r will be determined in equilibrium. The sequence of events is depicted in Figure 1.

2.1 Credit risk

We build on the conditional probability of default approach to credit risk based on systematic risk factors (see, e.g., Gordy (2003); Frey and McNeil (2003)). Specifically, credit risk is governed by $N+1$ random variables $\Theta_1 ... \Theta_N$, Z and a function $p(\theta, z)$ such that borrower i's probability of default is $p(\theta_i, z)$ whenever Θ_i and Z have taken on values θ_i and z respectively.[5] We call Z the *systematic risk factor*, $\Theta_1 ... \Theta_N$ the borrower-specific *frailty factors*, and impose the following assumptions.

- The frailty factors $\Theta_1 ... \Theta_N$ are independent, identically distributed, and independent of the systematic risk factor Z. The common distribution of the frailty factors is $\Pi(\theta)$ with density $\pi(\theta)$, and the distribution of the systematic risk factor is $G(z)$ with density $g(z)$.

- The number of borrowers is large ($N \to \infty$) so that, by a law of large numbers, the fraction of defaults among borrowers with frailty θ is almost surely $p(\theta, z)$ whenever the systematic risk factor Z has taken the value of z.

The support of Θ_i, $\text{supp}(\Theta_i) = \{\theta : \pi(\theta) > 0\}$ and the support of Z, $\text{supp}(Z) = \{z : g(z) > 0\}$ are both intervals given by (θ_1, θ_2) and (z_1, z_2) respectively. Thus, when the systematic risk factor Z has taken a value of $Z = z$, the probability of default for any given borrower will be

$$p(z) = \int_{\theta_1}^{\theta_2} p(\theta, z) d\Pi(\theta). \tag{1}$$

[5]We neglect that the probability of borrower default generally depends on the loan interest rate since, as we show in Section 6, this simplification streamlines the analysis without affecting our main results.

The unconditional default probability of any given borrower is then $\int_{z_1}^{z_2} p(z)dG(z)$. The borrower-specific frailty factors Θ_1 ... Θ_N and the systematic risk factor Z are realized on date 1 after all loans have been loans issued. Borrower i with frailty θ_i then repays his loan with probability $p(\theta_i, z)$ or defaults with the complement probability. Notice that loan defaults will be correlated due to the systematic risk factor, but borrowers with different frailties will be heterogeneously exposed to it.

Our specification can capture reduced form and structural default models of credit risk (Gordy, 2003; Frey and McNeil, 2003). One can thus think of $p(\theta, z)$ as the probability that the firm's net worth falls below a default threshold when the firm's frailty is θ and the systematic risk factor has taken a value of z. One can also think of $1 - p(\theta, z)$, or any monotone transformation of it, as measuring the borrower's distance to default.[6]

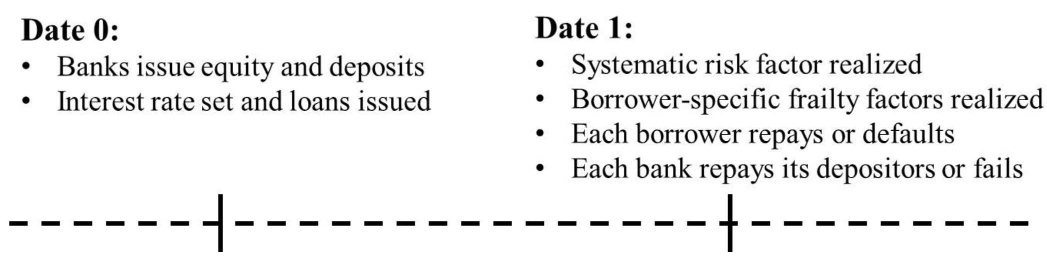

Figure 1: Timeline.

2.2 Banks

Banks issue loans on date 0 financed by a mix of equity and bank deposits. The banking sector is perfectly competitive, with no legacy assets, franchise values, or intermediation costs.[7] Deposits are insured, supplied perfectly elastically, at a net interest rate normalized to zero. Consider a bank issuing \$1 of loans financed with \$$k$ of capital and \$$1-k$ of deposits. Since all borrowers are identical when loans are issued, banking competition ensures they all get the same loan interest rate of r. The bank thus receives $1 + r$ from each borrower that repays his loan on date 1 but only $1 - \triangle$ from each borrower that defaults where $\triangle \in (0, 1]$ is the bank's loss-given loan default. The bank's net worth for a given realization of the

[6]For example, the conditional PD function can take the form $p(\theta, z) = F(\phi(z)|\theta)$ where $\phi(z)$ is some strictly increasing function in z and $\{F(x|\theta) : \theta \in [\theta_1, \theta_2]\}$ is a family of cumulative distribution functions indexed by the frailty parameter θ.

[7]Assuming perfect competition is to simplify the analysis but is not important otherwise, as we show in Section 6.

systematic risk factor $Z = z$ is then

$$y(z) = \underbrace{(1-p(z))(1+r)}_{\text{income from performing loans}} + \underbrace{p(z)(1-\triangle)}_{\text{income from non-performing loans}} - \underbrace{1-k}_{\text{amount promised to depositors}} \tag{2}$$

where $p(z)$ was defined in (1) and is the ratio of non-performing relative to all loans in the bank's portfolio. The bank fails on date 1 whenever its net worth turns negative $y(z) < 0$, which is equivalent to $p(z) > \frac{r+k}{r+\triangle}$. Letting q denote the probability of bank failure, we have

$$q = P\left\{z : p(z) > \tfrac{r+k}{r+\triangle}\right\}. \tag{3}$$

Each bank operates in the best interest of its shareholders, who are risk-neutral, protected by limited liability, and require an expected rate of return on their capital of at least $\delta > 0$. The parameter δ captures the scarcity of bank capital due to agency costs, equity issuance costs, the tax advantage of debt, or the liquidity premium of deposits.[8] The present value of the expected payoff for the bank's shareholders net of their capital contribution is thus

$$\pi = -k + \tfrac{1}{1+\delta} \int_{z_1}^{z_2} \max\{y(z), 0\}\, dG(z). \tag{4}$$

Finally, there is a minimum mandatory capital ratio of $k_{min} \in (0,1)$ set by regulation. To make the model interesting, we impose the parameter condition $\triangle > k_{min}$.[9] Since deposits are insured at a flat rate and bank capital is privately costly $\delta > 0$, each bank operates with the minimum possible capital ratio, namely $k^* = k_{min}$. The equilibrium loan interest rate r^* would then be such that the bank's shareholders break even in expectation

$$\int_{z_1}^{z_2} \max\{(1-p(z))(1+r^*) + p(z)(1-\triangle) - (1-k_{min}), 0\}\, dG(z) = (1+\delta)k_{min}. \tag{5}$$

We will denote by $y^*(z)$ the equilibrium net worth of the bank as a function of z, which is given by the expression in (2) with $r = r^*$ and $k = k_{min}$. Finally, the equilibrium bank failure probability q^* is obtained by inserting the equilibrium values of the loan interest rate and capital ratio, that is, r^* and k_{mn}, into the expression for the bank's failure probability in (3).

[8]See, e.g., Holmstrom and Tirole (1997) and Diamond and Rajan (2001) for models micro-founding a positive value of δ. It should be noted that our result does not hinge on assuming privately costly bank capital $\delta > 0$. Even if $\delta = 0$, it is well-known that the bank would still strictly prefer to operate with the minimum possible capital ratio whenever it can fail with a positive probability since this allows it to exploit the deposit insurance guarantee (Repullo, 2004; Bahaj and Malherbe, 2020).

[9]This is necessary since $\triangle \leq k_{min}$ implies $p(z) \leq \frac{r+k}{r+\triangle}$ for all z. As a result, the bank always has a sufficient capital buffer to absorb loan losses and cannot fail.

In summary, an economy is defined by (i) a conditional PD function $p(\theta, z)$, (ii) a distribution Π for the frailty factors Θ_1 ... Θ_N and a distribution G for the systematic risk factor Z, (iii) the bank's loss given loan default $\triangle \in [0, 1)$, (iv) a minimum required rate of return for the bank's shareholders $\delta > 0$, and (v) a minimum capital ratio $k_{min} \in (0, 1)$.

Finally, even though the subsequent analysis is carried out in terms of bank failure, it should be noted that, in the spirit of *value-at-risk*, one can equivalently investigate the probability that the realized loss on a given portfolio of loans consisting of ex-ante homogeneous borrowers exceeds a given pre-specified target level (see Section 6).

2.3 Hazard rate functions

It will be useful to first recall the standard definition of a hazard rate. Let T be the lifetime of an item with distribution function $F(t) = P\{T \leq t\}$, survival function $\bar{F}(t) = 1 - F(t)$, and density function $f(t)$. The hazard fate function (also called the failure rate function) is defined as $h(t) = f(t)/\bar{F}(t)$ for $t \geq 0$. In other words, $h(t)dt$ is the infinitesimal conditional probability that the item will fail in the next dt units of time, given that it has survived t units of time. Next, let $F(t|\theta) = P\{T \leq t|\theta\}$ be a family of distributions indexed by a parameter $\theta \in \Theta$. Assume each subpopulation $F(t|\theta)$ has density $f(t|\theta)$. One then mixes these subpopulations according to some probability distribution $\Pi(.)$ on Θ leading to a mixture density $f(t) = \int_\Theta f(t|\theta)d\Pi(\theta)$ and a mixture survival function $\bar{F}(t) = \int_\Theta \bar{F}(t|\theta)d\Pi(\theta)$. Then $h(t) = f(t)/\bar{F}(t)$ is the mixture hazard rate.

Time is not the only risk factor, and, as is well-known, the hazard rate function $h(t)$ can be defined with respect to any variable - it is not even necessary for this variable to be non-negative (Shaked and Shanthikumar, 2007). The hazard rate functions in our case will be defined with respect to the systematic risk factor Z, whereas the mixing variable θ will be borrower frailty. Specifically, the hazard rate function $h(z|\theta)$ associated with the conditional PD function $p(\theta, z)$ is

$$h(z|\theta) = \tfrac{d}{dz}[-\log(1 - p(\theta, z))] = \frac{\frac{\partial}{\partial z}p(\theta,z)}{1-p(\theta,z)}, \tag{6}$$

where the borrower frailty θ is treated as a parameter. The hazard rate function has a natural interpretation. If the systematic risk factor has taken a value of $Z = z$, the proportion of solvent borrowers among those with frailty θ will be $1 - p(z|\theta)$. Thus, $h(z|\theta)dz$ is the proportion of those borrowers that default given an infinitesimal increase dz in the systematic risk factor. The function $p(\theta, z)$ is said to have an increasing (decreasing) hazard rate if $h(z|\theta)$ is monotonically increasing (decreasing) in z. On the other hand, $p(\theta, z)$ has a constant hazard rate if $h(z|\theta)$ is constant in z, which is the case if and only if $p(\theta, z) = e^{-\theta\phi(z)}$ where

$\phi(z)$ is some linear function in z. The mixture hazard rate $h(z)$ associated with $p(z)$ is defined as

$$h(z) = \frac{p'(z)}{1-p(z)} = \frac{\int_{\theta_1}^{\theta_2} \frac{\partial}{\partial z} p(\theta,z)\pi(\theta)d\theta}{\int_{\theta_1}^{\theta_2}(1-p(\theta,z))\pi(\theta)d\theta}. \tag{7}$$

In other words, the frailty factor is the mixing variable since it leads to different subpopulations of borrowers based on their frailty. We impose the following assumptions:

- The function $p(\theta, z)$ is strictly increasing and absolutely continuous in z.[10]

- The family of hazard rate functions $h(z|\theta)$, $\theta \in [\theta_1, \theta_2]$ are ordered in θ and, in particular, $h(z|\theta)$ is strictly increasing in θ.

3 Main results

We introduce a *baseline* economy and a *safer* economy as a laboratory to study how borrower credit quality shapes equilibrium bank risk. We assume throughout that the distribution of the frailty factor is the same in both economies $\Pi(\theta) = \tilde{\Pi}(\theta)$ for all $\theta \in (\theta_1, \theta_2)$ and denote its density by $\pi(\theta)$. We also take the remaining parameters ($\triangle$, k_{min} and δ) and the distribution of the systematic risk factor Z to be the same in both economies.[11] The ratio of non-performing loans as a function of the systematic risk factor is $p(z)$ in the baseline economy and $\tilde{p}(z)$ in the safer economy. That is,

$$\underbrace{p(z) = \int_{\theta_1}^{\theta_2} p(\theta,z)\pi(\theta)d\theta}_{\text{Baseline economy}} \qquad \text{and} \qquad \underbrace{\tilde{p}(z) = \int_{\theta_1}^{\theta_2} \tilde{p}(\theta,z)\pi(\theta)d\theta}_{\text{Safer economy}}, \tag{8}$$

where $p(\theta, z)$ is the conditional PD function in the baseline economy and $\tilde{p}(\theta, z)$ is the conditional PD function in the safer economy. From (3), the equilibrium probability of bank failure in the baseline economy q^* and in the safer economy $\tilde{q}^*$ will be

$$q^* = P\left\{p(z) > \tfrac{r^*+k_{min}}{r^*+\triangle}\right\} \qquad \text{and} \qquad \tilde{q}^* = P\left\{\tilde{p}(z) > \tfrac{\tilde{r}^*+k_{min}}{\tilde{r}^*+\triangle}\right\},$$

where r^* is the equilibrium loan interest rate in the baseline economy and $\tilde{r}^*$ is the equilibrium loan interest rate in the safer economy. In the above, we use that the bank in each economy

[10] We do not really need absolute continuity of $p(\theta, z)$ with respect to z (i.e., continuity will suffice) since we are primary interested in the hazard rate order (see, e.g., Section 1 in Shaked and Shanthikumar (2007)) but impose it nevertheless in keeping with the usual approach in survival analysis.

[11] Section 6 outlines one way of micro-founding a common systemic risk factor based on a well-known property of first-order stochastic dominance.

operates with the minimum possible capital ratio k_{min}, which, moreover, is the same for both economies.

3.1 Unconventional outcome and risk-reversals

The unconventional outcome happens whenever the bank in the baseline economy is *less* likely to fail than the bank in the safer economy even though the conditional PD function in the baseline economy is *always* higher than in the safer economy, That is, $p(\theta, z) > \tilde{p}(\theta, z)$ for all θ and all z.

Definition 1. *The* unconventional outcome *happens whenever:*
(i) $p(\theta, z) > \tilde{p}(\theta, z)$ *for all* $\theta \in (\theta_1, \theta_2)$ *and all* $z \in (z_1, z_2)$ *and*
(ii) $q^* < \tilde{q}^*$.

Definition 1 implies $p(z) > \tilde{p}(z)$ for all $z \in (z_1, z_2)$. To rule out some uninteresting cases, we will further impose

$$\lim_{z \to z_2} p(\theta, z) = 1 \qquad \text{for all } \theta \in (\theta_1, \theta_2). \tag{9}$$

This implies that the bank in the baseline economy fails with some probability greater than zero, and it is necessary to rule out uninteresting situations where the bank in the 'safer' economy is more likely to fail simply due to *greater* systematic risk.[12] Next, we define risk-reversal. In particular, let $q^*(\Pi)$ and $\tilde{q}^*(\Pi)$ denote the probability of bank failure in the baseline and safer economy, respectively, when the distribution of the frailty factor in each economy is Π. Also, let δ_θ denote a degenerate distribution at θ. That is, $\delta_\theta(x) = 0$ if $x < \theta$ and $\delta_\theta(x) = 1$ if $x \geq \theta$.

Definition 2. Risk-reversal *happens whenever: (i)* $q^*(\delta_\theta) \geq \tilde{q}^*(\delta_\theta)$ *for all* $\theta \in (\theta_1, \theta_2)$ *and*
(ii) $q^*(\Pi) < \tilde{q}^*(\Pi)$ *for some distribution function* $\Pi : [\theta_1, \theta_2] \to [0, 1]$.

Risk reversal happens whenever the bank in the baseline economy is *more* likely to fail when all borrowers in the two economies have the same frailty but *less* likely when the borrowers differ in their frailty. Stated differently, risk reversal implies that the outcome will be unconventional, but only if the dispersion of the frailty factor is large enough.

[12] For example, assume that each borrower in the baseline economy defaults with probability 0.1, but defaults are independent for all z. As a result, the bank in the baseline economy never fails. On the other hand, each borrower in the safer economy defaults with a probability of 0.01, but defaults are perfectly correlated: for z large enough, all borrowers default, and the bank fails. Notice that the conditional PD function in the baseline economy, in that case *does not* satisfy (9).

3.2 Sufficient conditions

We now provide sufficient conditions for the unconventional outcome and risk reversals in Proposition 1 in terms of the hazard rate properties of the conditional PD functions $p(\theta, z)$ and $\tilde{p}(\theta, z)$. Denote by $\tilde{h}(z|\theta)$ and $\tilde{h}(z)$ the hazard rate function and the mixture hazard rate function associated with the conditional PD function in the safer economy $\tilde{p}(\theta, z)$. Consider the following relationships:

$$h(z|\theta) \geq \tilde{h}(z|\theta) \qquad \text{for all } z \in (z_1, z_2) \text{ and all } \theta \in (\theta_1, \theta_2) \tag{10}$$

$$h(z) < \tilde{h}(z) \qquad \text{for all } z \in (z_1, z_2) \tag{11}$$

It will be useful to keep mind that $h(z|\theta) \geq \tilde{h}(z|\theta)$ for all z is equivalent to $(1-p(\theta, z))/(1-\tilde{p}(\theta, z))$ weakly decreasing in z and similarly, $h(z) \geq \tilde{h}(z)$ for all z is equivalent to $(1-p(z))/(1-\tilde{p}(z))$ weakly decreasing in z. The conditions in (10) - (11) state that for each θ the component hazard rate in the baseline economy $h(z|\theta)$ is (weakly) higher than the component hazard rate in the safer economy $\tilde{h}(z|\theta)$. At the same time, the mixture hazard rate in the baseline economy $h(z)$ is strictly lower than the mixture hazard rate in the safer economy $\tilde{h}(z)$. This is possible since, as is well-known, the hazard rate order is not closed under mixtures (Barlow and Proschan, 1975).[13]

Proposition 1. *Suppose $p(\theta, z) > \tilde{p}(\theta, z)$ for all $\theta \in (\theta_1, \theta_2)$ and all $z \in (z_1, z_2)$. Then*

(i) The unconventional outcome happens whenever (11) holds. That is, $h(z) < \tilde{h}(z)$ for all $z \in (z_1, z_2)$.

(ii) Risk-reversal happens whenever (10) and (11) hold. That is, $h(z|\theta) \geq \tilde{h}(z|\theta)$ and $h(z) < \tilde{h}(z)$ for all $z \in (z_1, z_2)$ and all $\theta \in (\theta_1, \theta_2)$.

The proof of this proposition is in the appendix, whereas Section 3.3 provides intuition. Several remarks are in order. First, the bank in the baseline economy will be less likely to fail than the bank in the safer economy whenever (11) holds regardless of whether, in addition, one also imposes $p(\theta, z) > \tilde{p}(\theta, z)$ for all θ and z. That is, the mixture hazard rate order is sufficient to rank the equilibrium probability of bank failure in the two economies. Second, we only need $h(z) < \tilde{h(z)}$ for all values of Z such that the bank in the baseline economy remains solvent. That is, $\{z : y^*(z) \geq 0\}$. The ranking of the hazard rates for other values of z is immaterial to whether the bank in the baseline economy is more or less likely to fail.

[13] We will investigate why this happens in Section 4. For now, we offer an example based on Block et al. (2003). Take $h(z|\theta) = \theta - e^{-5(z+1)}$ and $\tilde{h}(z|\theta) = \theta - e^{-5z}$ implying $h(z|\theta) > \tilde{h}(z|\theta)$ for all $z \in [0, \infty)$. Then, if the frailty factor for each borrower in each economy is either $\theta = 1$ or $\theta = 6$ with equal probability, we obtain a hazard rate reversal, namely $h(z) < \tilde{h}(z)$ for all $z \in [0, \infty)$.

3.3 Loan performance vs loan interest effect

The hazard rate ranking conveys useful information about how the probability of bank failure varies with borrower risk. In particular, to determine whether the bank in the baseline economy is more or less likely to fail than the bank in the safer economy, it is necessary to examine the relationship between the two banks' equilibrium net worth as a function of the systematic risk factor, as shown in Figure 2. Specifically, for each z, the equilibrium relationship between the net worth of the two banks can be decomposed into a loan performance and a loan interest effect, as shown below.

$$\underbrace{\tilde{y}^*(z)}_{\text{Safer economy}} = \underbrace{y^*(z)}_{\text{Baseline economy}} + \underbrace{(p(z) - \tilde{p}(z))(\tilde{r}^* + \triangle)}_{\text{Loan performance effect}} - \underbrace{(1 - p(z))(r^* - \tilde{r}^*)}_{\text{Loan interest effect}}. \tag{12}$$

The bank in the safer economy has a lower ratio of non-performing loans than the bank in the baseline economy $\tilde{p}(z) < p(z)$ and thus gains an income of $(p(z) - \tilde{p}(z))(\tilde{r}^* + \triangle)$ from those borrowers that default in the baseline economy but repay their loan in the safer economy (*loan performance effect*). Simultaneously, the loan interest rate in the safer economy is lower than in the baseline economy $\tilde{r}^* < r^*$ (see below), leading to the bank in that economy to lose an income of $(1 - p(z))(r^* - \tilde{r}^*)$ from those borrowers that would repay their loan in both economies (*loan interest effect*). Next, since the shareholders of each bank must earn the same expected equilibrium rate of return (equal to δ) the net worth functions $y^*(z)$ and $\tilde{y}^*(z)$ will cross at least once at some point $z' \in (z_1, z_2)$ whose value will be determined in equilibrium.[14] In particular, we obtain from (12) that the net worth functions are equal $y^*(z') = \tilde{y}^*(z')$ if and only if the equilibrium loan interest rate in the baseline economy r^* and the equilibrium loan interest rate in the safer economy $\tilde{r}^*$ satisfy the following relationship

$$r^* = \tilde{r}^* + \tfrac{p(z') - \tilde{p}(z')}{1 - p(z')}(\tilde{r}^* + \triangle). \tag{13}$$

Then, since $p(z) > \tilde{p}(z)$ for all z, including z', the above immediately implies that the loan interest rate will be higher in the baseline economy $r^* > \tilde{r}^*$. We then show in the proof of Proposition 1 that if the mixture hazard rates are ordered as in (7), namely $h(z) < \tilde{h}(z)$ for all z, then $y^*(z') = \tilde{y}^*(z')$ implies $\frac{dy^*(z')}{dz'} > \frac{d\tilde{y}^*(z')}{dz'}$. Hence, $y^*(z)$ will cross $y^*(z)$ only once and from below as shown in Figure 2 implying $y^*(z) < \tilde{y}^*(z)$ for all $z < z'$ and $y^*(z) > \tilde{y}^*(z)$ for all $z > z'$. The overall result is that the bank in the baseline economy will be *less* likely to fail than the bank in the safer economy.

[14] If such a point does not exist, the shareholders of the bank in one of the economies would earn strictly greater equilibrium expected return than the shareholders of the bank in the other economy, which is inconsistent with equilibrium.

Remarks. One can use the decomposition in (12) to weaken the sufficient conditions for the unconventional outcome in Proposition 1. In particular, the net worth of the bank in the baseline economy $y^*(z)$ equals the net worth of the bank in the safer economy $\tilde{y}^*(z)$ if and only if the loan performance effect equals the loan interest effect $(p(z)-\tilde{p}(z))(\tilde{r}^*+\triangle) = (1-p(z))(r^*-\tilde{r}^*)$. Denote by Z^* the set of all realizations of the systematic risk factor such that these two effects are equal. The bank in the baseline economy is then less likely to fail (i.e., the unconventional outcome happens) whenever $h(z) < \tilde{h}(z)$ for all $z \in Z^*$.

Also, it is not hard to generalize the above argument to situations where the two economies are distinct in terms of minimum capital ratio, loan-loss given default, and the interest rate that must be promised to the depositors. We also imposed perfect competition and assumed the bank's shareholders in each bank earn the same expected return, but this is not needed either, nor is it necessary to assume that the probability of borrower default does not depend on the loan interest rate. All these modifications of the base setup are considered in Section 6.

3.4 An example

The following example features both the unconventional outcome and a risk reversal and illustrates how neglecting borrower heterogeneity can be misleading. The frailty for each borrower in each economy is either $\theta = 0.1$ or $\theta = 0.5$ and the conditional PD function in the baseline economy $p(\theta, z)$ and in the safer $\tilde{p}(\theta, z)$ economy are given by

$$p(\theta, z) = 1 - e^{-\theta(z+1)^{1.1}} \qquad \text{and} \qquad \tilde{p}(\theta, z) = 1 - e^{-\theta z^{1.1}}, \tag{14}$$

where the support of the systematic risk factor is $(0, \infty)$.[15] Notice that $p(\theta, z) > \tilde{p}(\theta, z)$ for all θ and all z and $\lim_{z\to\infty} p(\theta, z) = 1$. Figure 3 displays $\frac{\tilde{q}^*-q^*}{q^*}$ as a function of the probability of the low frailty realization $\eta \equiv P\{\Theta = 0.1\}$. Whether or not the unconventional outcome happens depends on the variance of the frailty factor, namely $\mathrm{Var}(\theta) = \eta(1-\eta)$. In particular, there are three distinct regions in the figure.

- (i) For low values of η, the bank in the baseline economy is more likely to fail than the bank in the safer economy $q^* > \tilde{q}^*$.
- (ii) For intermediate values of η, the bank in the baseline economy is less likely to fail than the bank in the safer economy $q^* < \tilde{q}^*$.

[15] The specification in (14) is a version of the proportional hazard model, which has a long history in credit risk modeling and is examined at length in Section 5.

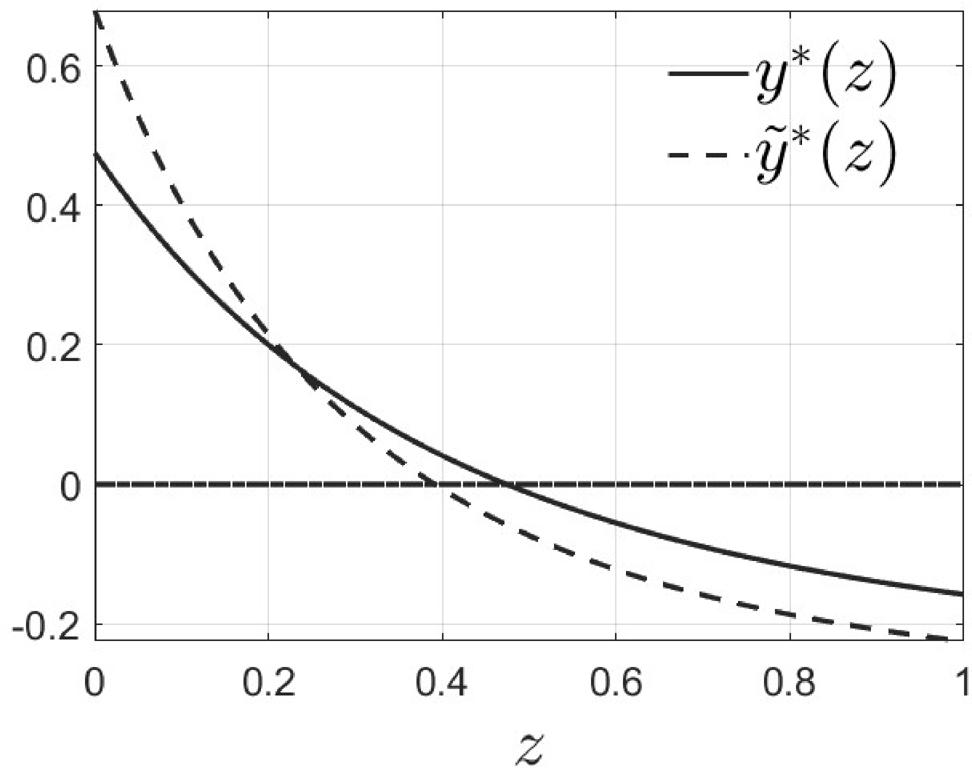

Figure 2: This figure displays the equilibrium net worth $y^*(z)$ ($\tilde{y}(z)$) of the bank in the baseline (safer) economy as a function of the realized value of the systematic risk factor $Z = z$. The relationship between $y^*(z)$ and $\tilde{y}(z)$ is given in (12), and it can be decomposed into a loan performance and a loan interest effect. The bank in each economy fails whenever its net worth is negative.

- (iii) For high values of η, the bank in the baseline economy is again more likely to fail than the bank in the safer economy $q^* > \tilde{q}^*$.

It is easy to see that this example also features risk-reversal as per Definition 2 since the bank in the baseline economy is more likely to fail whenever the frailty for all borrowers is fixed to $\theta = 0.5$, which corresponds to $\eta = 0$ in Figure 3. Similarly, the bank in the baseline economy is more likely to fail whenever the frailty for all borrowers is fixed to $\theta = 0.1$, which corresponds to $\eta = 1$. That is, mixing borrowers with different frailty can significantly alter and even reverse, as in this example, the relationship between borrower and bank risk. The next section investigates what makes borrower heterogeneity conducive to the unconventional outcome.

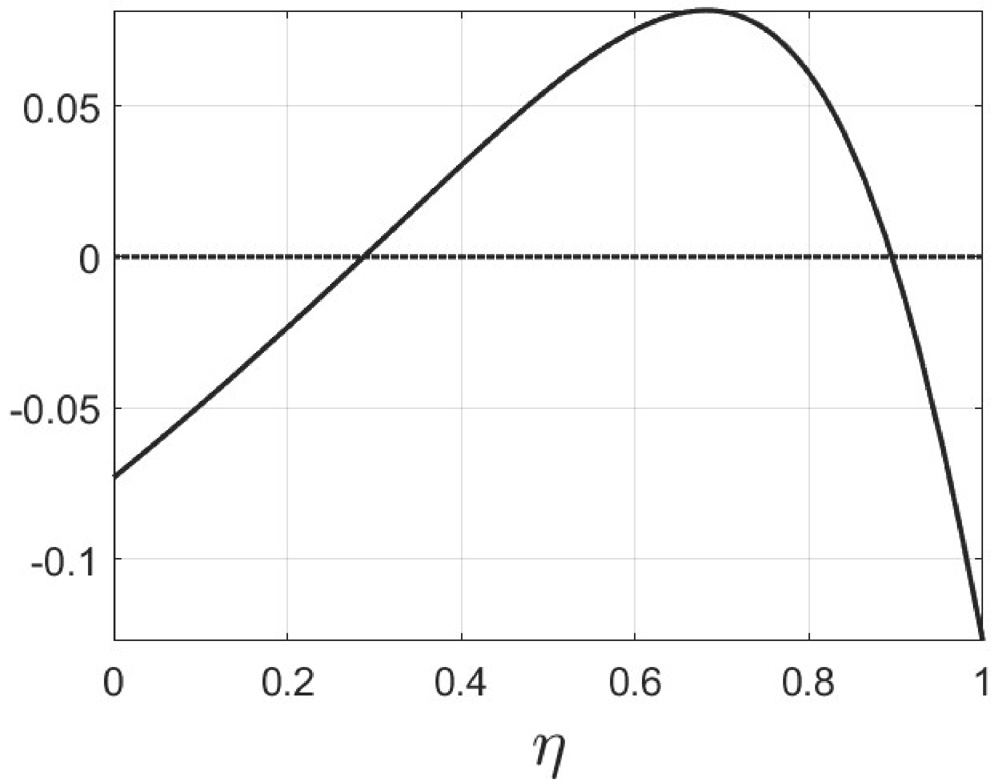

Figure 3: This figure is from Section 3.4 and displays $\frac{\tilde{q}^*-q^*}{q^*}$ as a function of the probability of the low frailty realization for each borrower $\eta \equiv P\{\Theta = 0.1\}$. The remaining parameters in each economy are $k_{min} = 0.05$, $\delta = 0.01$, and $\triangle = 0.3$. The distribution of the systematic risk factor in each economy is exponential $G(z) = 1 - e^{-1.5z}$. The unconventional outcome happens $\tilde{q}^* > q^*$ but only if the variance of the frailty factor, namely $\mathrm{Var}(\theta) = \eta(1-\eta)$, is large enough.

4 What leads to risk-reversals?

This section examines the conditions leading to risk reversals which, as shown in Proposition 1, are closely tied to *hazard rate reversals*, namely, situations such that mixture hazard rates behave very differently than their components. To fix ideas and illustrate the forces at play, we begin with a simple example. Then, we introduce a convenient decomposition of the mixture hazard rates that will be used to derive necessary, and in some cases sufficient, conditions for risk reversals.

$p(\theta, z)$ for the baseline economy	z_1	z_2	$\tilde{p}(\theta, z)$ for the safer economy	z_1	z_2
Low frailty θ_1	0.058	0.467	Low frailty θ_1	0.026	0.444
High frailty θ_2	0.262	0.956	High frailty θ_2	0.125	0.946

Table 1: Example of hazard rate reversal. Each table entry shows the probability of borrower default for a given combination of the frailty factor and the systematic risk factor in the respective economy. Each realization of the frailty factor is equally likely.

4.1 Hazard rate reversals - an example

Table 1 displays the conditional borrower PD in each economy where both variables are taken to be binary. That is, $\Theta \in \{\theta_1, \theta_2\}$, $Z \in \{z_1, z_2\}$. Moreover, each value of the frailty factor is equally likely $P\{\Theta = \theta_1\} = P\{\Theta = \theta_2\} = 0.5$. The mixture hazard rate in each economy can be decomposed as follows (the expressions below are a special case of (15) and (16)):

$$\underbrace{\tfrac{p(z_2)-p(z_1)}{1-p(z_1)}}_{0.656} = \underbrace{\left(\tfrac{p(\theta_1,z_2)-p(\theta_1,z_1)}{1-p(\theta_1,z_1)}\right)}_{0.434}\underbrace{\left[\tfrac{(1-p(\theta_1,z_1))\frac{1}{2}}{(1-p(\theta_1,z_1))\frac{1}{2}+(1-p(\theta_2,z_1))\frac{1}{2}}\right]}_{0.560} + \underbrace{\left(\tfrac{p(\theta_2,z_2)-p(\theta_2,z_1)}{1-p(\theta_2,z_1)}\right)}_{0.940}\underbrace{\left[\tfrac{(1-p(\theta_2,z_1))\frac{1}{2}}{(1-p(\theta_1,z_1))\frac{1}{2}+(1-p(\theta_2,z_1))\frac{1}{2}}\right]}_{0.440}$$

$$\underbrace{\tfrac{\tilde{p}(z_2)-\tilde{p}(z_1)}{1-\tilde{p}(z_1)}}_{0.670} = \underbrace{\left(\tfrac{\tilde{p}(\theta_1,z_2)-\tilde{p}(\theta_1,z_1)}{1-\tilde{p}(\theta_1,z_1)}\right)}_{0.429}\underbrace{\left[\tfrac{(1-\tilde{p}(\theta_1,z_1))\frac{1}{2}}{(1-\tilde{p}(\theta_1,z_1))\frac{1}{2}+(1-\tilde{p}(\theta_2,z_1))\frac{1}{2}}\right]}_{0.527} + \underbrace{\left(\tfrac{\tilde{p}(\theta_2,z_2)-\tilde{p}(\theta_2,z_1)}{1-\tilde{p}(\theta_2,z_1)}\right)}_{0.938}\underbrace{\left[\tfrac{(1-\tilde{p}(\theta_2,z_1))\frac{1}{2}}{(1-\tilde{p}(\theta_1,z_1))\frac{1}{2}+(1-\tilde{p}(\theta_2,z_1))\frac{1}{2}}\right]}_{0.473}$$

The hazard rate in the baseline economy is *higher* than the hazard rate in the safer economy for each value of the frailty factor (0.434 vs 0.429 and 0.940 vs 0.938). However, the mixture hazard rate is *lower* in the baseline economy (0.656 vs 0.670). There are two effects at play leading to this outcome. First, the default rate among the high-frailty borrowers is lower in the safer economy (0.125 vs. 0.262), implying that the proportion of high-frailty solvent borrowers relative to all solvent borrowers is higher in that economy (0.473 vs. 0.440). As a result, the mixture hazard rate in the safer economy is, to a greater extent, determined by the hazard rate of its high-frailty borrowers than the mixture hazard rate in the baseline economy. Second, the high-frailty borrowers in the safer economy still have a higher hazard rate than the low-frailty borrowers in the baseline economy (0.938 vs 0.434), and this last effect is strong enough to generate a hazard rate reversal in this particular case.

4.2 A useful decomposition of the mixture hazard rates

The previous example shows that hazard rate reversals can be traced to the changing composition of solvent borrowers as a function of the systematic risk factor. To systematically study these effects, we adopt the approach of Lynn and Singpurwalla (1997). In particular, the mixture hazard rate in (7) can be recast as a weighted sum of the corresponding component hazard rates in (6). That is,

$$h(z) = \int_{\theta_1}^{\theta_2} h(z|\theta)\pi(\theta|z)d\theta \qquad \text{where} \qquad \pi(\theta|z) \equiv \frac{(1-p(\theta,z))\pi(\theta)}{\int_{\theta_1}^{\theta_2}(1-p(\theta,z))\pi(\theta)d\theta}, \tag{15}$$

where recall that $\pi(\theta)$ is the density function of the frailty factor in the baseline economy. For z fixed, the conditional density function $\pi(\theta|z)$ is the ratio of solvent borrowers with frailty θ relative to all solvent borrowers.[16] That is, each component hazard rate $h(z|\theta)$ must be weighted by the remaining solvent borrowers with the corresponding frailty $(1 - p(\theta, z))\pi(\theta)$ relative to all remaining solvent borrowers $\int_{\theta_1}^{\theta_2}(1 - p(\theta, z))\pi(\theta)d\theta$. Thus, there are two opposing effects since high $p(\theta, z)$, and therefore low $1 - p(\theta, z)$, is associated with high $h(z|\theta)$ and vice versa. Similarly, the mixture hazard rate for the safer economy is

$$\tilde{h}(z) = \int_{\theta_1}^{\theta_2} \tilde{h}(z|\theta)\tilde{\pi}(\theta|z)d\theta \qquad \text{where} \qquad \tilde{\pi}(\theta|z) \equiv \frac{(1-\tilde{p}(\theta,z))\pi(\theta)}{\int_{\theta_1}^{\theta_2}(1-\tilde{p}(\theta,z))\pi(\theta)d\theta}. \tag{16}$$

From (15) and (16) it is not hard to see that we have $h(z) \geq \tilde{h}(z)$ for all z (ruling out the unconventional outcome and risk-reversals) whenever $h(\theta, z) \geq \tilde{h}(\theta', z)$ for all θ, θ' and all z. These, however, are rather strong conditions since they require even the lowest-frailty borrowers in the baseline economy to have a higher hazard rate than the highest-frailty borrowers in the safer economy.[17]

4.3 Weak-die-first-effect and its implications

By subtracting $\int_{\theta_1}^{\theta_2} \tilde{h}(z|\theta)\pi(\theta|z)d\theta$ from $h(z)$ and from $\tilde{h}(z)$ and using the expressions for the mixture hazard rates in (15) and (16) we obtain that $h(z) < \tilde{h}(z)$ is equivalent to

$$\int_{\theta_1}^{\theta_2} \left(h(z|\theta) - \tilde{h}(z|\theta)\right)\pi(\theta|z)d\theta < \underbrace{\int_{\theta_1}^{\theta_2} \tilde{h}(z|\theta)\left(\tilde{\pi}(\theta|z) - \pi(\theta|z)\right)d\theta}_{\text{Weak-die-first-effect}}. \tag{17}$$

[16] Observe that, since the hazard rate is a conditional characteristic, the ordinary expectation with respect to θ does not define a mixture hazard rate and proper conditioning must be used, namely that in (15).

[17] This is not the case in the Example from Section 4.1 where the high-frailty borrowers in the safer economy have a hazard rate of 0.938, which is considerably higher than the hazard rate of the low-frailty borrowers in the baseline economy 0.434.

We have used a term from survival analysis to label the right-hand side as the *weak-die-first effect* (Vaupel et al., 1979) or, in our case, the high-frailty-borrowers-default-first-effect. It should be noted that the weak-die-first-effect in the context of survival analysis describes how the composition of items in a given mixture changes with the passage of time, whereas in our case this effect is interpreted as describing the composition of solvent borrowers in the baseline relative to the safer economy as one varies the systematic risk factor Z.[18] Recall from Proposition 1 that risk-reversals happen whenever $h(z|\theta) \geq \tilde{h}(z|\theta)$ for all θ and z and $h(z) < \tilde{h}(z)$ for all z. Hence, risk reversals necessarily imply a weak-die-first-effect defined as the term on the right-hand side of (17) being positive.

Proposition 2. *Suppose* $\frac{1-\tilde{p}(\theta,z)}{1-p(\theta,z)}$ *is strictly increasing in* θ *for all* z*. Then there is a weak-die-first-effect. That is,*

$$\int_{\theta_1}^{\theta_2} \tilde{h}(z|\theta) \left(\tilde{\pi}(\theta|z) - \pi(\theta|z)\right) d\theta > 0 \qquad \textit{for all } z.$$

This proposition is based on a theorem of Finkelstein and Esaulova (2006), showing that if the mixing random variables are ordered in the sense of likelihood ratio, then the mixture hazard rates can be ordered as well. In essence, $\frac{1-\tilde{p}(\theta,z)}{1-p(\theta,z)}$ strictly increasing in θ allows one to implement change of variable and then order the mixing variable in the sense described above. Intuitively, $\frac{1-\tilde{p}(\theta,z)}{1-p(\theta,z)}$ increasing in θ implies that, for each z, the mixture of solvent borrowers in the baseline economy contains a lower fraction of high frailty borrowers than the mixture of solvent borrowers in the safer economy generating a weak-die-first-effect in the baseline relative to the safer economy.[19]

Corollary 1. *Suppose* $\frac{1-\tilde{p}(\theta,z)}{1-p(\theta,z)}$ *is strictly increasing in* θ *for all* z*.*

(i) The unconventional outcome happens whenever $h(z|\theta) \leq \tilde{h}(z|\theta)$ *for all* θ *and* z*.*

(ii) Risk-reversal happens whenever $h(z|\theta) = \tilde{h}(z|\theta)$ *for all* θ *and* z*.*

Next, assume the conditional PD functions satisfy the relationship $p(\theta, z) = \tilde{p}(\theta, \phi(z))$ for some strictly increasing function $\phi(z)$ such that $\psi(z) > z$ for all z.[20] It is not hard to see that the following will be true in this case

$$h(z|\theta) = \phi'(z)\tilde{h}(\phi(z)|\theta) \qquad \text{and} \qquad h(z) = \phi'(z)\tilde{h}(\phi(z)).$$

[18] Of course, only one values of the systematic risk factor will be realized on date 1 and by varying the realization of Z we mean a counterfactual exercise where one uses the credit risk model to examine properties of the mixture of solvent borrowers for different value of Z.

[19] It is easy to check that the example in Table 1 satisfies the condition in Proposition 2.

[20] Notice that this is with loss of generality as it assumes that the function $\phi(z)$ does not depend on θ.

As a result, the condition in (11), namely $h(z) < \tilde{h}(z)$ for all z, is equivalent to $\phi'(z)\tilde{h}(\phi(z)) < \tilde{h}(z)$ for all z. Moreover, if $\phi(z) = z + \alpha$ for some constant $\alpha > 0$, then $h(z) < \tilde{h}(z)$ is equivalent to the mixture hazard rate in the safer economy $\tilde{h}(z)$ strictly decreasing in z. This simple derivation, and a well-known property of mixture hazard rates, leads to another corollary of Proposition 2.

Corollary 2. *Suppose $p(\theta, z) = \tilde{p}(\theta, \phi(z))$ where the function $\phi(z)$ is strictly increasing, differentiable, and such that $\phi(z) > z$. Then the unconventional outcome happens whenever $\phi'(z)\tilde{h}(\phi(z)) < \tilde{h}(z)$ holds for all z. If, in addition, $\phi(z) = z + \alpha$ for some $\alpha > 0$. Then*

(i) The unconventional outcome happens whenever $\tilde{h}(z|\theta)$ is weakly decreasing in z for all θ.

(ii) Risk-reversal happens whenever $\tilde{h}(z|\theta)$ is constant in θ for all z.

This corollary stems from the well-known fact that $\tilde{h}(z|\theta)$ weakly decreasing in z for each θ implies that the mixture hazard rate $\tilde{h}(z)$ is strictly decreasing in z for any non-degenerate distribution for the frailty factor (Barlow and Proschan, 1975). For completeness, the appendix contains proof of this important result, which can be linked to properties of log-convex functions.

5 Application: proportional hazard models

We now apply the analysis from the previous sections to the *proportional hazard* model, which is of interest due to its wide use in credit risk modeling (Duffie and Singleton, 2003) and other fields (Finkelstein, 2008). Let the conditional probability of default in the baseline economy $p(\theta, z)$ and in the safer economy $\tilde{p}(\theta, z)$ be as follows

$$p(\theta, z) = 1 - e^{-\theta\Lambda(\phi(z))} \qquad \text{and} \qquad 1 - \tilde{p}(\theta, z) = e^{-\theta\Lambda(z)}, \tag{18}$$

where $z \geq 0$ and $\theta \geq 0$. The function $\phi(z)$ governs how credit risk in the two economies relates. It is strictly increasing, differentiable, and such that $\phi(z) > z$ for all z. The function $\Lambda(z)$ (to be precise, its first derivative $\Lambda'(z)$) is called the *base hazard rate* and it is non-negative, differentiable, and strictly increasing.[21]

Proposition 3. *Suppose the conditional PD functions $p(\theta, z)$ and $\tilde{p}(\theta, z)$ are as in (18) and the distribution of the frailty factor is the same in the two economies with a moment-generating function $M_\Theta(t) = \int_{\theta_1}^{\theta_2} e^{-\theta t} d\Pi(\theta)$. Then*

[21] The example in Section 3.4 was a special case with $\phi(z) = z + \alpha$ and $\Lambda(z) = (z + \gamma)^\beta$ where γ and α are parameters. Also, note that $\phi(z)$ independent of θ implies that credit risk in both economies is governed by a proportional hazard model with the base hazard rate in the baseline economy defined as $H(z) \equiv \Lambda(\phi(z))$.

(i) The unconventional outcome happens whenever $\frac{d}{dz}\frac{M_\Theta(-\Lambda(\phi(z)))}{M_\Theta(-\Lambda(z))} > 0$.

(ii) Risk-reversal happens whenever $\frac{d}{dz}\frac{M_\Theta(-\Lambda(\phi(z)))}{M_\Theta(-\Lambda(z))} > 0$ *and* $\phi'(z)\Lambda'(\phi(z)) \geq \Lambda'(z)$ *for all* z.

One tractable variant of the proportional hazard model is to assume that the frailty factor is gamma-distributed with density function

$$\pi(\theta) = \frac{1}{\Gamma(a)b^a}\theta^{a-1}e^{-\theta/b}, \tag{19}$$

where a and b are the shape and scale parameters. Since frailty cannot be negative in the proportional hazard model, the gamma distribution, along with the log-normal and Weibull distribution, is a natural choice. The gamma distribution is very flexible since it can take on a variety of shapes, becoming the exponential distribution for $a = 1$, whereas, for large a, it assumes a bell shape similar to the normal distribution. The moment-generating function of the gamma distribution is $M_\Theta(t) = e^{-a\log(1+bt)}$. Proposition 3 then implies that the unconventional outcome happens whenever

$$\frac{d}{dz}\frac{1+b\Lambda(z)}{1+b\Lambda(\phi(z))} > 0. \tag{20}$$

If, in addition, one assumes that the function $\phi(z)$ is such that $\phi(z) = z + \alpha$ for some parameter $\alpha > 0$ the above condition for the unconventional outcome would simplify to

$$\frac{\Lambda''(z)}{(\Lambda'(z))^2} < \frac{b}{1+\Lambda(z)}, \tag{21}$$

which is equivalent to the condition derived by Gurland and Sethuraman (1995) for a decreasing mixture hazard rate in this type of model. This is not a surprise since if $\phi(z) = z+\alpha$ then $h(z) < \tilde{h}(z)$ for all z is equivalent to $\tilde{h}(z)$ strictly decreasing in z.

An example. Let $\phi(z) = z + \alpha$ and $\Lambda(z) = (z+\gamma)^\beta$ where $\gamma > 0$ and $\alpha > 0$ are parameters and suppose the frailty factor in each economy is gamma distributed with density as in (19). Then the unconventional outcome happens whenever $\tilde{h}(z)$ is strictly decreasing in z, which is true whenever the parameter values are such that

$$\gamma^\beta > (\beta - 1)b.$$

Notice that the above will be satisfied for all $\beta \in (0, 1]$. On the other hand, risk-reversal happens whenever $h(z|\theta) \geq \tilde{h}(z|\theta)$ for all θ and z and $h(z) < \tilde{h}(z)$ for all z which is true whenever the parameter values are such that

$$\gamma^{\beta} > (\beta - 1)b \geq 0.$$

The difference with the previous condition is that $(\beta - 1)b \geq 0$ implies $\beta \geq 1$. Finally, it should be noted that Corollary 1 implies that in the borderline case $\beta = 1$, the outcome is unconventional *and* features risk-reversal for *any* non-degenerate distribution for the frailty factor and not only for the gamma distribution.

Weak-die-first-effect. One other notable feature of the proportional hazard model in (18) is that it always features a weak-die-first-effect (see Section 4) since:

$$\tfrac{1-\tilde{p}(\theta,z)}{1-p(\theta,z)} = e^{\theta(\Lambda(\phi(z))-\Lambda(z))},$$

which, under the maintained assumptions on $\Lambda(z)$ and $\phi(z)$, is strictly increasing in z. Even if a proportional hazard model does not exactly govern credit risk in the baseline economy but instead we have

$$p(\theta, z) = 1 - e^{-\theta\Lambda(\phi(\theta,z))},$$

where $\phi(\theta, z)$ now also depends on θ, one can still derive relatively straightforward conditions for the weak-die-first-effect. In particular, $\frac{1-\tilde{p}(\theta,z)}{1-p(\theta,z)}$ will be strictly increasing in θ if and only if

$$\Lambda(\phi(z)) - \Lambda(z) + \theta\Lambda'(\phi(\theta, z))\tfrac{\partial\phi(\theta,z)}{\partial\theta} > 0.$$

The above is satisfied whenever $\frac{\partial\phi(\theta,z)}{\partial\theta} \geq 0$. In other words, if $\frac{\partial\phi(\theta,z)}{\partial\theta}$ is non-decreasing in θ then the proportional hazard setup implies that for each z the mixture of solvent borrowers in the safer economy is to a greater extent skewed towards high frailty borrowers than the mixture of solvent borrowers in the baseline economy which, as we saw in Proposition 2, is a necessary condition for risk-reversals.

6 Discussion

Here, we clarify the role of the assumptions and show that our main results are robust to several natural variations of the baseline setup.

Role of Competition

Perfect banking competition is *not* necessary for our results, and, in fact, one can remain agnostic about the exact nature of banking competition and intermediation costs in the two economies as long as they share a *common* systematic risk factor Z. That is, given any equilibrium loan interest rates r^* and $\tilde{r}^*$ (however these were actually determined), the equilibrium net worth of the two banks would still satisfy the relationship in (12).[22] In other words, r^* and $\tilde{r}^*$ will be related as in (13) regardless of the nature of banking competition. The analysis then proceeds verbatim, and Proposition 1 applies as long as the equilibrium net worth of the two banks crosses at least once for some value of Z, such that both banks are solvent, as shown in Figure 2. In other words, as long as we rule out uninteresting cases where one of the banks is less likely to fail simply because it is more profitable than the other bank for each realization of Z.

Systematic risk factor

We assumed the two economies have the *same* systematic risk factor Z. Here, we outline one reason this could be the case. Suppose the conditional PD function in the baseline economy is $p_B(\theta, y_B)$ where y_B is the realization of the systematic risk factor Y_B and the conditional PD function in the safer economy is $p_S(\theta, y_S)$ where y_S is the realization of another systematic risk factor Y_S. Assume $p_B(\theta, y_B)$ is strictly increasing in y_B and $p_S(\theta, y_S)$ is strictly increasing in y_S. Further, assume $Y_B \geq_1 Y_S$ where $\geq_1$ denotes first-order stochastic dominance.[23] Hence, there exists another random variable Z and functions ϕ_B and ϕ_S such that $\phi_B(z) \geq \phi_S(z)$ for all z and $Y_B =_D \phi_B(Z)$ and $Y_S =_D \phi_S(Z)$ where $=_D$ denotes equality in distribution. This property, known as *Strassen's theorem*, actually goes both ways: $Y_B \geq_1 Y_S$ if and only if there is a random variable Z and functions ϕ_B and ϕ_S such that $\phi_B(z) \geq \phi_S(z)$ for all z and $Y_B =_D \phi_B(Z)$ and $Y_S =_D \phi_S(Z)$ (see, e.g., Section 1 in Shaked and Shanthikumar (2007)). Then take Z to be the systematic risk factor in each economy and define the conditional PD functions $p(\theta, z) \equiv p_B(\theta, \phi_B(z))$ and $\tilde{p}(\theta, z) \equiv p_S(\theta, \phi_S(z))$.

Borrower PD depends on r

The probability of borrower default was assumed to be independent of the loan interest rate, but our main result does not hinge on this assumption. Suppose the conditional PD function in the baseline economy is $p(\theta, z, r)$ and the conditional PD function in the safer economy is $\tilde{p}(\theta, z, r)$. Then define the functions $p^*(\theta, z) \equiv p(\theta, z, r^*)$ and $\tilde{p}^*(\theta, z) \equiv \tilde{p}(\theta, z, \tilde{r}^*)$ where, as before, r^* is the equilibrium loan interest rate in the baseline economy and $\tilde{r}^*$ the equilibrium

[22]We maintain the assumption that deposits are insured at a flat-rate and perfectly elastically supplied at an interest rate which is the same in the two economies and normalized to zero.

[23]If X and Y are two random variables such that $P\{X > x\} \geq P\{Y > x\}$ for all $x \in (-\infty, \infty)$ then X is said to dominate Y in the first-order stochastic sense denoted $X \geq_1 Y$. Intuitively, X is more likely than Y to take on larger values.

loan interest rate in the safer economy. Further, suppose that due to credit rationing or moral hazard (we do not need to take a stance on the exact reason), the probability of borrower default is increasing in the loan interest rate. The analysis proceed exactly as before but with $p(\theta, z)$ and $\tilde{p}(\theta, z)$ replaced by their equilibrium counterparts $p^*(\theta, z)$ and $\tilde{p}^*(\theta, z)$ respectively.

Discrete Z

If the systematic risk factor is discrete, the unconventional outcome will arise under similar circumstances. In fact, one can derive sufficient and necessary conditions in the special case of a binary systematic risk factor.

Proposition 4. *Suppose the systematic risk factor is binary $Z \in \{z_1, z_2\}$ where $z_1 < z_2$. Assume $p(z_1) > \tilde{p}(z_1)$ and $p(z_2) > \tilde{p}(z_2)$. Let $\eta = P\{Z = z_2\}$ denote the probability of the bad state. The unconventional outcome then happens if and only if*

$$\frac{p(z_2)-p(z_1)}{1-p(z_1)} \leq \frac{(1+\delta)k_{min}}{(\eta+\delta)k_{min}+(1-\delta)\triangle} < \frac{\tilde{p}(z_2)-\tilde{p}(z_1)}{1-\tilde{p}(z_1)}. \tag{22}$$

For example: $p(z_1) = 0.2$, $p(z_2) = 0.3$, $\tilde{p}(z_1) = 0.05$, $\tilde{p}(z_2) = 0.25$, $\eta = 0.05$, $\triangle = 0.3$, $k_{min} = 0.05$, and $\delta = 0.01$. The equilibrium loan interest rate in the baseline economy is 0.0780, the equilibrium loan interest rate in the safer economy is 0.0191, and *only* the bank in the baseline economy fails in the bad state, thus leading to the unconventional outcome.

Difference along several dimensions

The only distinctive feature of the baseline relative to the safer economy was the conditional PD function, but this was for simplicity (and to identify exactly the effect of borrower risk) since we can allow the two economies to differ along multiple dimensions. Assume the loan-loss given default is $\triangle$ in the baseline economy and $\tilde{\triangle}$ in the safer economy, the minimum capital ratio is k_{min} in the baseline economy and $\tilde{k}_{min}$ in the safer economy, and the equilibrium deposit interest rate is r_D^* in the baseline economy and $\tilde{r}_D^*$ in the safer economy. Again, we can remain agnostic about the exact reason for these differences and whether they are due to regulation, market forces, or both. The bank in the baseline economy is then *less* likely to fail than the bank in the safer economy, leading to the unconventional outcome whenever the mixture hazard rates satisfy

$$h(z) < \tilde{h}(z) \left[1 - \frac{\triangle - \tilde{\triangle} + (1-k_{mn})(1+r_D^*) - (1-\tilde{k}_{min})(1+\tilde{r}_D^*)}{(1-p(z))(r^* + \triangle(z))}\right] \qquad \text{for all } z. \tag{23}$$

The above reduces to (11) whenever $k_{min} = \tilde{k}_{min}$, $r_D^* = \tilde{r}_D^*$, and $\triangle = \tilde{\triangle}$. On the other hand, the term in square brackets in (23) is less than one - pushing against the unconventional outcome - whenever the bank in the baseline economy has greater loss given loan default $\triangle > \tilde{\triangle}$, or it must pay more to its depositors $r_D^* > \tilde{r}_D^*$, or it must operate with higher minimum capital ratio $k_{min} > \tilde{k}_{min}$ than the bank in the safer economy.

Government guarantees

We assumed throughout that deposits are insured, leading to the question of whether what we find can be attributed to frictions due to miss-priced government guarantees, which are well-known for distorting banks' choices along several dimensions.[24]. Even if that were the case, our analysis would still be useful since banks *are* exposed to distortionary government guarantees. However, one can show that miss-priced deposit insurance is not driving the results since the takeaway from Figure 3 remains the same under fairly priced deposit insurance premiums. In other words, the channels emerge independently from distortions due to miss-priced government guarantees.

Portfolio loss reformulation

Instead of casting the analysis in terms of bank failure, one can investigate the probability that the loss on a given portfolio of loans would exceed a pre-specified level, say k. In particular, if $L(z) \equiv -(y(z)-1)$ is the realized loss on the loan portfolio where $y(z)$ is given in (2), then the probability of the event $\{z : L(z) > k\}$ is the same as in (3). The only thing that changes is the interpretation of q, which now is the probability that the portfolio loss exceeds the pre-specified level, with the rest of the analysis proceeding verbatim. This reformulation is useful for the following: the bank has J different types of borrowers on date 0 indexed by $j \in \{1, ..., J\}$ (in the baseline model $J = 1$). The conditional default probability for the type-j borrower is $p_j(\theta, z)$ where θ and z are as before, but now we index the conditional PD function with the borrower type. The bank then forms J different portfolios - one for each borrower type - and examines the probability of the event $\{z : L_j(z) > k\}$ where $L_j(z)$ is the loss on the j-th portfolio and k_j is a pre-specified tolerance level for that particular portfolio.

7 Conclusion

The conventional view is that, other things equal, a bank with safer borrowers will be safer than a bank with riskier borrowers. As a result, banks with safer borrowers should not be as heavily scrutinized by regulators and allowed to hold less capital or, more generally, reduce

[24]See, e.g., Kareken and Wallace (1978); Repullo (2004); Harris et al. (2020); Bahaj and Malherbe (2020)

the risk mitigation they undertake compared to banks with riskier borrowers. This paper uses tools from survival analysis to challenge this view by showing that the bank with the safer borrowers can, nevertheless, be more likely to fail than the bank with the riskier borrowers. Further, this outcome tends to happen under greater borrower heterogeneity and cannot be attributed to miss-priced government guarantees or a particular competitive environment. Instead, the mechanism behind this outcome is traced to the changing composition of solvent borrowers as a function of the systematic risk factor, illustrating how borrower heterogeneity can significantly alter the equilibrium relationship between borrower and bank risk.

Appendix

Proof of Proposition 1.

It remains to show that $h(z) < \tilde{h}(z)$ for all z implies the slope of $y^*(z)$ is strictly greater than the slope of $\tilde{y}^*(z)$ at any point, say z', such that $y^*(z') = \tilde{y}^*(z')$. That is, we want to show

$$y^*(z') = \tilde{y}^*(z') \qquad \Rightarrow \qquad \tfrac{dy^*(z')}{dz'} > \tfrac{d\tilde{y}^*(z')}{dz'}$$

Recall from (2) that the net worth of each bank is given by

$$y^*(z) = (1 - p(z))(1 + r^*) + p(z)\triangle - (1 - k_{min})$$

$$\tilde{y}^*(z) = (1 - \tilde{p}(z))(1 + \tilde{r}^*) + \tilde{p}(z)\triangle - (1 - k_{min})$$

Hence, $y^*(z') = \tilde{y}^*(z')$ is equivalent to

$$r^* - p(z')(r^* + \triangle) = \tilde{r}^* - \tilde{p}(z')(\tilde{r}^* + \triangle)$$

or adding $\triangle$ to both sides

$$(1 - p(z'))(r^* + \triangle) = (1 - \tilde{p}(z'))(\tilde{r}^* + \triangle) \tag{24}$$

Differentiating the net worth functions $y^*(z)$ and $\tilde{y}^*(z)$ with respect to z:

$$\tfrac{dy^*(z)}{dz} = -p'(z)(r^* + \triangle) = -\tfrac{p'(z)}{1-p(z)}(1 - p(z))(r^* + \triangle)$$

$$\tfrac{d\tilde{y}^*(z)}{dz} = -\tilde{p}'(z)(\tilde{r}^* + \triangle) = -\tfrac{\tilde{p}'(z)}{1-\tilde{p}(z)}(1 - \tilde{p}(z))(\tilde{r}^* + \triangle)$$

Finally, evaluating these derivatives at $z = z'$ and using the relationship in (24) yields

$$\frac{dy^*(z')}{dz'} > \frac{d\tilde{y}^*(z')}{dz'} \qquad \Leftrightarrow \qquad \frac{p'(z')}{1-p(z')} = h(z') < \frac{\tilde{p}'(z')}{1-\tilde{p}(z')} = \tilde{h}(z') \tag{25}$$

But since $h(z) < \tilde{h}(z)$ for all z we get $\frac{dy^*(z')}{dz'} > \frac{d\tilde{y}^*(z')}{dz'}$ as desired.

Proof of Proposition 2.

Suppose $\frac{1-\tilde{p}(\theta,z)}{1-p(\theta,z)}$ is strictly increasing in θ for all z. We must show the right-hand side of (17) is positive, which is equivalent to showing

$$\int_{\theta_1}^{\theta_2} \tilde{h}(z|\theta)\tilde{\pi}(\theta|z)d\theta > \int_{\theta_1}^{\theta_2} \tilde{h}(z|\theta)\pi(\theta|z)d\theta \tag{26}$$

Denote by $\Pi(\theta|z) = \int_{\theta_1}^{\theta} \pi(\theta|z)d\theta$ and $\tilde{\Pi}(\theta|z) = \int_{\theta_1}^{\theta} \tilde{\pi}(\theta|z)d\theta$ the distribution functions corresponding to $\pi(\theta|z)$ and $\tilde{\pi}(\theta|z)$. Since $\tilde{h}(z|\theta)$ is assumed to be strictly increasing in θ it will be sufficient to show $\tilde{\Pi}(.|z) \geq_1 \Pi(.|z)$ where $\geq_1$ denotes *first-order stochastic dominance*. Consider the following:

$$\begin{aligned}
\tilde{\pi}(\theta|z) \quad &= \quad \frac{(1-\tilde{p}(\theta,z))\pi(\theta)}{\int_{\theta_1}^{\theta_2}(1-\tilde{p}(\theta,z))\pi(\theta)d\theta} \quad = \quad \frac{(1-p(\theta,z))\left[\frac{1}{A(z)}\frac{1-\tilde{p}(\theta,z)}{1-p(\theta,z)}\pi(\theta)\right]}{\int_{\theta_1}^{\theta_2}(1-p(\theta,z))\left[\frac{1}{A(z)}\frac{1-\tilde{p}(\theta,z)}{1-p(\theta,z)}\pi(\theta)\right]d\theta} \\
&= \quad \frac{(1-p(\theta,z))m(\theta|z)}{\int_{\theta_1}^{\theta_2}(1-p(\theta,z))m(\theta|z)d\theta}
\end{aligned}$$

where $m(\theta|z)$ is a probability density function defined as

$$m(\theta|z) \equiv \frac{1}{A(z)}\frac{1-\tilde{p}(\theta,z)}{1-p(\theta,z)}\pi(\theta)$$

and $A(z) \equiv \int_{\theta_1}^{\theta_2} \frac{1-\tilde{p}(\theta,z)}{1-p(\theta,z)}\pi(\theta)d\theta$ is a normalizing factor ensuring $m(\theta|z)$ is a proper density (i.e., $\int_{\theta_1}^{\theta_2} m(\theta|z)d\theta = 1$). Then $\frac{1-\tilde{p}(\theta,z)}{1-p(\theta,z)}$ strictly increasing in θ implies the ratio $\frac{m(\theta|z)}{\pi(\theta)}$ is strictly increasing in θ. That is,

$$\frac{d}{d\theta}\left[\frac{m(\theta|z)}{\pi(\theta)}\right] = \frac{d}{d\theta}\left[\frac{1}{A(z)}\frac{1-\tilde{p}(\theta,z)}{1-p(\theta,z)}\right] > 0$$

Therefore, for all z, the density function $m(.|z)$ dominates the density function $\pi(.)$ in the *likelihood ratio sense*. Theorem 3 in Finkelstein and Esaulova (2006) then implies $\tilde{\Pi}(.|z) \geq_1 \Pi(.|z)$. Finally, since $\tilde{\Pi}(.|z) \geq_1 \Pi(.|z)$ and the distributions $\tilde{\Pi}(.|z)$ and $\Pi(.|z)$ are not the same Theorem 1.A.8. in Shaked and Shanthikumar (2007) implies that the strict inequality in (26) holds.

Proof of Corollary 2.

This corollary is based on the following closure under mixtures theorem of Barlow and Proschan (1975): if $\tilde{h}(z|\theta)$ is decreasing in z for all θ, then $\tilde{h}(z)$ is decreasing in z. For completeness, we provide proof based on the properties of log-convexity where, recall, that the function $1 - p(\theta, z)$ is *log-convex* in z means that $\log(1 - p(\theta, z))$ is convex in z. In particular, start from

$$1 - \tilde{p}(z) = \int_{\theta_1}^{\theta_2} (1 - \tilde{p}(\theta, z))\pi(\theta) d\theta$$

and suppose $\frac{d\tilde{h}(z|\theta)}{dz} \leq 0$ for all θ. Recall the definition of a hazard rate function:

$$\tilde{h}(z|\theta) = \tfrac{d}{dz}[-\log(1 - \tilde{\mathrm{p}}(\theta, \mathrm{z}))]$$

Hence $\frac{d\tilde{h}(z|\theta)}{dz} \leq 0$ if and only if $-\log(1 - \tilde{p}(\theta, z))$ is concave in z for θ or equivalently if and only if $1 - \tilde{p}(\theta, z)$ is log-convex in z for all θ. But then, since log-convexity is preserved under mixtures (Bagnoli and Bergstrom, 2006) $1 - \tilde{p}(z)$ will be log-convex. Therefore the mixture hazard rate $\tilde{h}(z)$ decreases in z. That is,

$$\tfrac{d\tilde{h}(z)}{dz} = \tfrac{d^2}{d^2 z}[-\log(1 - \tilde{p}(z))] \leq 0$$

The above inequality is strict whenever the distribution of the frailty factor $\Pi(\theta)$ is non-degenerate, as we assume to be the case.

Proof of Proposition 3.

The hazard rate function in the baseline $h(z|\theta)$ and in the safer economy $\tilde{h}(z|\theta)$ are given by

$$h(z|\theta) = \theta\phi'(z)\Lambda'(\phi(z)) \qquad \text{and} \qquad \tilde{h}(z|\theta) = \theta\Lambda'(z)$$

It follows that $h(z|\theta) \geq \tilde{h}(z|\theta)$ if and only if

$$\phi'(z)\Lambda'(\phi(z)) \geq \Lambda'(z) \tag{27}$$

Denoting by $M_\Theta(t) = \int_{\theta_1}^{\theta_2} e^{-\theta t} d\Pi(\theta)$ the moment-generating function for the frailty factor we get

$$1 - p(z) = \int_{\theta_1}^{\theta_2} (1 - p(\theta, z)) d\Pi(\theta) = M_\Theta(-\Lambda(\phi(z)))$$

for the baseline economy. Similarly, we get

$$1 - \tilde{p}(z) = M_\Theta(-\Lambda(z))$$

for the safer economy. Finally, recall that $h(z) < \tilde{h}(z)$ for all z is equivalent to $(1-p(z))/(1-\tilde{p}(z))$ strictly increasing in z (Shaked and Shanthikumar, 2007), which in that case is the same as

$$\tfrac{d}{dz}\tfrac{M_\Theta(-\Lambda(\phi(z)))}{M_\Theta(-\Lambda(z))} > 0 \tag{28}$$

Proposition 1 then implies that the unconventional outcome happens whenever (28) holds, whereas a risk-reversal happens whenever (27) and (28) jointly hold.

Proof of Proposition 4.

First, observe that

$$\tfrac{p(z_2)-p(z_1)}{1-p(z_1)} = 1 - \tfrac{1-p(z_2)}{1-p(z_1)} \qquad \text{and} \qquad \tfrac{\tilde{p}(z_2)-\tilde{p}(z_1)}{1-\tilde{p}(z_1)} = 1 - \tfrac{1-\tilde{p}(z_2)}{1-\tilde{p}(z_1)}$$

Hence, (22) will be equivalent to

$$\tfrac{1-p(z_2)}{1-p(z_1)} \geq \tfrac{\triangle - k_{min}}{\left(\frac{\eta+\delta}{1-\eta}\right)k_{min}+\triangle} > \tfrac{1-\tilde{p}(z_2)}{1-\tilde{p}(z_1)}$$

It will be sufficient to show that any bank fails in the bad state (i.e., when $Z = z_2$) if and only if the following holds

$$\tfrac{1-p(z_2)}{1-p(z_1)} < \tfrac{\triangle - k_{min}}{\left(\frac{\eta+\delta}{1-\eta}\right)k_{min}+\triangle} \tag{29}$$

Indeed, if the bank fails in the bad state and r^* is the equilibrium interest rate, then:

$$(1-\eta)\left[(1-p(z_1))(1+r^*) + p(z_1)(1-\triangle) - (1-k_{min})\right]$$

$$\text{and} \quad (1-p(z_2))(1+r^*) + p(z_2)(1-\triangle) < 1 - k_{min} \tag{30}$$

The first condition states that the bank's shareholders break even in expectation, whereas the second is that their bank fails in the bad state $Z = z_2$. Solving for the equilibrium loan interest rate r^*:

$$r^* = \tfrac{1}{1-p(z_1)}\left[\left(\tfrac{\eta+\delta}{1-\eta}\right)k_{min} + \triangle p(z_1)\right] \tag{31}$$

The above expression for r^* is valid if and only if the bank fails when the state is bad $Z = z_2$. We next show this is the case if and only if (29) holds. First, adding $\triangle$ to both sides of (31) yields

$$r^* + \triangle = \tfrac{1}{1-p(z_1)} \left[\left(\tfrac{\eta+\delta}{1-\eta} \right) k_{min} + \triangle \right] \tag{32}$$

Second, (30) is equivalent to

$$(r^* + \triangle)(1 - p(z_2)) < \triangle - k_{min}$$

Finally, using (32) to substitute for $r^* + \triangle$ in the above leads to the desired condition in (29).

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