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Abstract

We present a theoretical framework to characterize how financial market participants contribute to systemic risk, allowing us to derive optimal corrective policy interventions. To that end, we embed belief heterogeneity in a model of frictional financial markets. We document the asymmetry that, by their behavior, relatively more optimistic agents contribute more strongly to financial distress than more pessimistic agents do. We further show that financial distress is generally more likely in an economy whose agents hold heterogeneous rather than homogeneous beliefs. Based on these findings, we propose a system of non-linear Pigouvian taxes as the optimal corrective policy, which proves to generate considerable welfare gains over the linear policy advocated by former studies.

Key words: Financial amplification, pecuniary externalities, collateral constraint, financial crisis, belief heterogeneity, macroprudential policy

JEL codes: D84, E44, G28, H23

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1 Introduction

Systemic risk has been well studied since the global financial crisis. An important question remains yet to be explored: How can individual financial market participants' contributions to system-wide financial distress be measured, and how can they be addressed accordingly by Pigouvian policies?

The literature has provided valuable insights into the matter of measuring systemic risk. In particular, various approaches to specifying the financial system's exposure to certain institutions' risk taking have been suggested (Acharya et al., 2012; Adrian and Brunnermeier, 2016; Acharya et al., 2017). However, they cannot causally attribute the extent to which individual financial decisions, and the resulting marginal effects on market prices, contribute to systemic distress. Corrective policies thus lack a basis to be designed on.

In this paper, we build a theoretical framework to analyze contributions to systemic risk and optimal corrective policy interventions. We augment established models of financial frictions by heterogeneity of beliefs across the population, giving rise to differentiated risk taking in financial decisions. The latter are observable, so we may characterize distress contributions explicitly.

We find that economic agents make asymmetric contributions to financial distress, with more optimistic agents making larger contributions than pessimistic agents. We further show that financial distress is generally more likely in an economy whose agents hold heterogeneous rather than homogeneous and rational beliefs. The optimal policy we propose is a system of non-linear Pigouvian taxes, which proves to generate considerable welfare gains over the linear policy advocated by previous studies.

To the best of our knowledge, we are the first to combine a model of frictional financial markets with belief heterogeneity embedded in a single framework. Specifically, our model incorporates a price-dependent collateral constraint on borrowing. It introduces a pecuniary externalitiy, as economic agents do not internalize that their decisions mutually affect their borrowing capacities, which, in turn, establishes a financial amplification mechanism. Agents may hold heterogeneous beliefs in the sense of perceiving differentiated probability distributions over the future state of the world. This setup allows us to distinguish relatively more optimistic from pessimistic individuals.

We use this model to analyze the interaction of the financial friction and belief heterogeneity. First, we characterize how the latter impacts the probability of distress in the competitive equilibrium, as well as the equilibrium allocation, collateral prices, and externalities. We then perform an efficiency analysis, showing how a constrained social planner can attain a welfare improvement compared to the competitive equilibrium. We characterize her optimal corrective policies numerically, and evaluate how they influence social welfare and the probability of financial distress.

The analysis produces three key results. First, we find an asymmetry between optimistic and pessimistic agents' contributions to financial distress, attributing a stronger impact to the former. Moreover, optimistic agents prove to put downward pressure on collateral prices, tightening financial constraints. Under reasonable assumptions on the distribution of beliefs across the population, we conclude that belief heterogeneity precipitates financial distress.

Second, we show that, compared to an economy where agents hold a homogeneous and rational belief, belief heterogeneity raises the likelihood of financial distress. The reason is that, for collateral constraints to be binding, no sharp exogeneous shock to aggregate investment or net worth is required. Such a shock is typically assumed in the literature. Instead, it suffices that some agents' beliefs deviate from the ex post state of the world.

Third, we prove that, even though beliefs are agents' private information, a constrained social planner is able to establish a welfare improvement by means of a non-linear Pigouvian tax system. Our policy proposal contrasts the linear Pigouvian taxation proposed by previous studies. We provide numercial applications suggesting that our non-linear approach produces welfare gains relative to linear policies, and reduces the probability of financial distress.

This paper makes several important contributions. It provides a formal framework which can be used for further analyses of financial amplification mechanisms in environments where economic agents do not have rational expectations, but potentially feature heterogeneous beliefs. This helps to explicitly characterize how different market participants contribute to financial crises. Moreover, this lays the ground for an optimal design of prudential policies. Policy proposals not accounting for belief divergences might have only limited success if beliefs vary widely across financial market participants, as they cannot account for their respective contributions to systemic risk. This is particularly relevant during different phases of the business cycle, as investors' beliefs prove to fluctuate and diverge largely between booms and busts (Aliber and Kindleberger, 2015; Minsky, 1986; Kaplan et al., 2020; Mian and Sufi, 2022).

The remainder of this paper is organized as follows. We review the related literature in section 2. Section 3 develops the baseline model, and analyzes the competitive equilibrium. In section 4, we describe the externalities present in our model, derive optimal corrective policies, and perform normative analyses numerically. We provide some final remarks in section 5.

2 Related literature

Financial amplification and pecuniary externalities. Our model combines two strands of literature. First, it relates to the literature on financial amplification, including studies of pecuniary externalities in particular. This literature originates from Fisher (1933), and was extended by analyses of borrowing constraints and their effects on asset prices by Bernanke and Gertler (1990), Shleifer and Vishny (1992), Kiyotaki and Moore (1997), Brunnermeier and Pedersen (2009), and Acharya et al. (2011). Hart (1975) and Stiglitz (1982) moreover prove the presence of pecuniary externalities in incomplete markets.[1]

Welfare implications of pecuniary externalities are examined in Gromb and Vayanos (2002), Caballero and Krishnamurthy (2003), Lorenzoni (2008), and Caballero and Lorenzoni (2014). While these papers focus on externalities affecting borrowers' net worth, Jeanne and Korinek (2010), Bianchi (2011), Bianchi and Mendoza (2018), Dávila and Korinek (2018), and Jeanne and Korinek (2019) explore the collateral channel of financial amplification that can lead to financial crises. Since we are modeling externalities equivalently, we adopt their terminology and basic model structure.

Furthermore, we derive optimal corrective policies implemented by a a constrained social planner, referring to the early contributions of Stiglitz (1982) and Geanakoplos and Polemarchakis (1986). The policy maker in our model applies an ex-ante Pigouvian tax along the lines of Jeanne and Korinek (2010), Dávila and Korinek (2018), Jeanne and Korinek (2019), and Jeanne and Korinek (2020).[2]

In the domain of the aforementioned literature on financial market externalities, this paper links to articles that focus on defining measures of systemic risk. Notably, Adrian and Brunnermeier (2016) propose $\Delta CoVar$, a measure capturing the interdependences between specific financial institutions and the entire financial system. Furthermore, Acharya et al. (2012) and Acharya et al. (2017) model individual institutions' exposure to financial crises. For an overview of quantitative measures of systemic risk, see Bisias et al. (2012).

Macroeconomic perspectives on belief heterogeneity. The second strand of the literature relates to macroeconomic perspectives on belief heterogeneity. The idea of belief heterogeneity shaping market outcomes was pioneered by Keynes (1936), Minsky (1977), and Aliber and Kindleberger (2015). Since then, the literature has provided evidence that belief heterogeneity is relevant for asset prices and market volatility, in particular during the recent financial crisis (Harrison and Kreps, 1978; Scheinkman and Xiong, 2003;

[1]For survey articles, see Shleifer and Vishny (2011) and Brunnermeier and Oehmke (2013).

[2]The social planner in our model has an instrument at hand which could be interpreted as a financial transaction tax. So the interested reader is referred to the literature on financial transaction taxes initiated by Tobin (1978), and extended by Summers and Summers (1989) and Stiglitz (1989).

Reinhart and Rogoff, 2008; Simsek, 2013; Cheng et al., 2014; Gennaioli and Shleifer, 2018; Adam and Nagel, 2022).

As in all normative studies involving heterogeneity of beliefs, we face the challenge of how to aggregate welfare properly. Several approaches have been suggested, such as the welfare criteria put forth by Gilboa et al. (2014), Gayer et al. (2014), Brunnermeier et al. (2014), Blume et al. (2018), and Kim and Kim (2021).

Prior research has already combined the two strands of literature, particularly in the context of heterogeneous beliefs and leveraged speculation.[3] Geanakoplos (1996) was the first to model a general equilibrium with endogenous collateral constraints and heterogeneous beliefs, which was further developed in subsequent studies (Geanakoplos, 2003, 2010), showing that heterogeneity of beliefs fosters credit and leverage cycles. Simsek (2013) generalizes the framework, and focuses on various degrees of heterogeneity.

Belief heterogeneity and business cycles. Furthermore, our analysis is associated with the literature on business cycles. In particular, we refer to the role of beliefs in booms and busts. Minsky (1977, 1986) and Aliber and Kindleberger (2015) show how asset price booms are linked to increasing optimism. Rising asset prices create states of 'mania' in which investors are overly optimistic and hold the asset as they strongly believe that prices will continue to rise. Adam et al. (2017) argue that shifts in investors beliefs about future capital gains are highly relevant in explaining cyclical asset price fluctuations. Kaplan et al. (2020) show that such shifts were the driving force of the house-price boom prior to the global financial crisis. Moreover, Mian and Sufi (2022) elaborate on how important increasing divergence of beliefs was in the build-up of the house price boom prior to the 2007-2008 financial crisis.

Methodological approach. Lastly, our investigation of comparative statics with respect to the economy's belief structure closely relates to Dávila and Walther (2023), who study optimal leverage policies in response to changing beliefs. We follow their approach of applying methods from the calculus of variation to equilibrium variables under belief heterogeneity.

3 Model

The aim of this chapter is to explore financial amplification mechanisms in an environment where agents hold heterogeneous beliefs about the future. To that end, we set up a model

[3] Xiong (2013) and Simsek (2021) review the literature on asset trading driven by heterogeneous beliefs in great detail.

featuring frictional financial markets, and enrich it by belief heterogeneity across agents. We derive the competitive equilibrium of this economy, and study how it is impacted by variations in beliefs. The framework allows us to distinguish the respective contributions of optimistic and pessimistic agents to financial amplification, and to evaluate the probability of distress in economies with different belief structures. Our results lay the ground for the study of optimal Pigouvian policies in the next section.

3.1 Setup

We develop a model of a small open economy with three periods $t = 0, 1, 2$, and two classes of agents, called lenders and investors. Lenders trade debt securities with investors, or save in a zero return storage technology. The interest rate is exogeneous and normalized to zero for simplicity, and lenders are assumed to be risk-neutral.

Investors are divided into J groups indexed by $j \in \{1, ...J\}$, each of which consists of a continuum of investors. Each group has a population share s^j, that is common knowledge, and derives utility from a single consumption good c_t^j according to a concave and strictly increasing utility function $u\left(c_t^j\right)$. Population shares are collected in the vector $s = \{s^j\}_{j \in \{1,...J\}}$.

In $t = 0$, investors receive an endowment $e > 0$, as well as an initial amount of assets $\bar{a} > 0$. They can borrow or save d_0^j to finance consumption, and to further invest into a_0^j units of the asset.[4] The asset is traded at a price q_0, and exists in fixed supply.

In $t = 1$, financial investment pays off an a priori uncertain dividend $R \in \left[\underline{R}, \overline{R}\right]$, which different groups of investors hold specific beliefs about. After all uncertainty has been resolved at the beginning of the period, investors repay former debt d_0^j, issue new debt d_1^j, and trade again, purchasing or liquidating l_1^j claims on the asset at price q_1.

Debt issuance in $t = 1$ is restricted by a borrowing constraint

$$d_1^j \leq \phi q_1 \left(a_0^j - l_1^j\right).$$

The constraint implies that investors borrow against their asset position at the end of the period.[5]

In $t = 2$, net of claims $a_0^j - l_1^j$ materializes and debt d_1^j must be repaid, determining final consumption c_2^j.

Our model features two important components. First, financial markets exhibit a fric-

[4] Lenders' endowment is assumed to make the supply of debt securities perfectly elastic to demand. That is, all investors' borrowing preferences can be satisfied by assumption. This includes the possibility of savings $d_0^j < 0$.

[5] For a better intuition of the micro-mechanism underlying this constraint, see Jeanne and Korinek (2019).

tion, captured by the the borrowing constraint. It incorporates a financial amplification mechanism within our framework, and results in a pecuniary externality. Second, we allow investors to hold different beliefs about the asset pay-off R.

Definition 1. *Let $F(R)$ be the true cumulative distribution function (cdf) of R, and $F^j(R)$ be the cdf perceived by type-j investors. We refer to heterogeneous beliefs if each type of investors j perceives an idiosyncratic distribution of R, i.e. $F^i(R) \neq F^j(R)$ for all $i \neq j$. We refer to homogeneous beliefs if all types of investors have rational expectations, i.e. $F^j(R) = F(R)$ for all j.*

The vector $\mathcal{F} = \{F^j(R)\}_{j\in\{1,\dots J\}}$ characterizes the complete set of beliefs existing in the economy, which is publicly known. Beliefs are distributed discretely across types, so each cdf $F^j(R)$ appears with frequency s^j.

3.2 Competitive equilibrium

To derive the competitive equilibrium, we first conduct individual optimization backwards from $t = 2$ to $t = 0$. We distinguish between state variables of type-j individuals, i.e. $\{a_0^j, d_0^j\}$, and aggregate state variables of group j, denoted by $\left\{\tilde{a}_0^j, \tilde{d}_0^j\right\}$.

Optimization in $\mathbf{t = 1, 2}$. The optimization problem of type-j investors in $t = 1$ reads

$$V_1^j\left(a_0^j, d_0^j|\tilde{a}_0, \tilde{d}_0\right) = \max_{c_1^j, c_2^j, d_1^j, l_1^j \leq a_0^j} u\left(c_1^j\right) + u\left(c_2^j\right) \quad \text{s.t.}$$

$$\left(\lambda_1^j\right) \quad c_1^j = Ra_0^j + q_1 l_1^j + d_1^j - d_0^j \tag{1}$$

$$\left(\lambda_2^j\right) \quad c_2^j = a_0^j - l_1^j - d_1^j \tag{2}$$

$$\left(\eta_1^j\right) \quad d_1^j \leq \phi q_1\left(a_0^j - l_1^j\right), \tag{3}$$

where investors take group-wide aggregate states $\tilde{a}_0 = \left\{\tilde{a}_0^j\right\}_{j\in\{1,\dots J\}}$ and $\tilde{d}_0 = \left\{\tilde{d}_0^j\right\}_{j\in\{1,\dots J\}}$ as given because they affect the equilibrium asset price q_1. Let λ_1^j and λ_2^j be the Lagrange multipliers for the budget constraints (1) and (2), respectively, and η_1^j for the borrowing constraint (3).

This problem produces the following Euler equations for each j:

$$u'\left(c_1^j\right) - \eta_1^j = u'\left(c_2^j\right) \tag{4}$$

$$q_1 u'\left(c_1^j\right) - \eta_1^j \phi q_1 = u'\left(c_2^j\right), \tag{5}$$

jointly yielding equilibrium price equations

$$q_1 = \frac{u'\left(c_2^j\right)}{(1-\phi)u'\left(c_1^j\right) + \phi u'\left(c_2^j\right)} \tag{6}$$

for each j.

Optimization in t = 0. In $t = 0$, the optimization of a type-j investor is

$$\max_{c_0^j, a_0^j \geq 0, d_0^j} \quad u\left(c_0^j\right) + E^j\left[V_1^j\left(a_0^j, d_0^j | \tilde{a}_0, \tilde{d}_0\right)\right] \quad \text{s.t.}$$
$$\left(\lambda_0^j\right) \quad c_0^j = e + d_0^j + q_0\left(\bar{a} - a_0^j\right), \tag{7}$$

where the expectation operator is indexed by j, capturing potentially differing beliefs, and λ_0^j denotes the Lagrange multiplier for the period-0 budget constraint. Eliminating Lagrange multipliers, we obtain the following two optimality conditions:

$$q_0 u'\left(c_0^j\right) = E^j\left[Ru'\left(c_1^j\right) + u'\left(c_2^j\right) + \eta_1^j \phi q_1\right] \tag{8}$$
$$u'\left(c_0^j\right) = E^j\left[u'\left(c_1^j\right)\right]. \tag{9}$$

Equilibrium. In equilibrium, the asset market is cleared in both periods $t = 0$ and $t = 1$, formalized by the conditions

$$\sum_{j=1}^{J} s^j a_0^j = \bar{a} \tag{10}$$

and

$$\sum_{j=1}^{J} s^j l_1^j = 0. \tag{11}$$

Complementing the optimality conditions derived thus far, they complete the set of equilibrium conditions. In a symmetric equilibrium, investors are identical within each group j, i.e. $x_t^j = \tilde{x}_t^j$ for all j with $x \in \{c, a, d, l, \lambda, \eta\}$. We may thus define the symmetric competitive equilibrium as follows.

Definition 2. *A competitive equilibrium consists of an allocation* $\left\{\tilde{c}_0^j, \tilde{c}_1^j, \tilde{c}_2^j, \tilde{a}_0^j, \tilde{d}_0^j, \tilde{d}_1^j, \tilde{l}_1^j\right\}_{j \in \{1, \ldots J\}}$*, a sequence of multipliers* $\tilde{\eta}_1 = \left\{\tilde{\eta}_1^j\right\}_{j \in \{1, \ldots J\}}$*, and prices* $\{q_0, q_1\}$*, satisfying equations* (1), (2), (4), (5), (7), (8), (9), *and a complementary slackness condition for all* j*, as well as the market clearing conditions* (10) *and* (11)*, given population shares* s *and beliefs* $\mathcal{F}$*.*

The competitive equilibrium reflects the two main components of our model: the financial friction and potential belief disagreements. The financial friction introduces a wedge between market prices of the asset as well as debt, and investors' marginal rates of substitution across periods. The wedge is formally represented by the multiplier $\tilde{\eta}_1^j$ that appears in equations (4), (5), and (8). In the latter two equations, the term $\tilde{\eta}_1^j \phi q_1$ captures the collateral premium of the asset, as each additional unit of $\tilde{a}_0^j$ and $\tilde{l}_1^j$ relaxes the constraint.

To highlight the impact of belief heterogeneity, we compare the competitive equilibrium under heterogeneous and homogeneous beliefs. If investors have heterogeneous expectations of the return R, they evaluate expected marginal benefits of investment and borrowing differently. Formally, group-specific expectation operators E^j apply in the Euler equations (8) and (9), resulting in group-specific values of $\tilde{a}_0^j$, $\tilde{d}_0^j$, and of the shadow price of borrowing $\tilde{\eta}_1^j$.

If, in contrast, investors hold a homogeneous belief, their marginal rates of substitution are identical, as is the shadow value of borrowing. Importantly, intertemporal substitution in this case is only possible through debt or savings $\tilde{d}_t^j$. The reason is that investors do not trade in excess of the initial asset endowment neither in $t = 0$ nor in $t = 1$, i.e. $\tilde{a}_0^j = \bar{a}$ and $\tilde{l}_1^j = 0$ for all j.

In the following, we restrict the set of equilibria taken into account in the analysis. Since we are only interested in situations when financial distress occurs, the model parameters, comprising risk aversion A, beliefs $\mathcal{F}$, the realized return $\hat{R}$, as well as the margin requirement ϕ, must satisfy that, in equilibrium, the asset is traded and constraints are binding ($\tilde{\eta}_1^j > 0$).[6]

Period-1 equilibrium price. Given its impact on the borrowing constraint, the equilibrium collateral price q_1 is a key variable in our model. We show its existence and uniqueness, and how it interacts with the multiplier of the borrowing constraint.

Proposition 1.

(i) The equilibrium price q_1 exists.

(ii) If at least one type of investors j receives a return as expected or higher, i.e. $E^j[R] \leq \hat{R}$ for at least one j and any realization $\hat{R}$ of R, the equilibrium price is unique, satisfying $q_1 \leq 1$, and the following two equivalences hold:

(1) $q_1 = 1$ iff $\tilde{\eta}_1^j = 0$ for all j

(2) $q_1 < 1$ iff $\tilde{\eta}_1^j > 0$ for at least one j.

[6]We make parameter restrictions explicit in the derivations of our results, provided in the appendix.

Proposition 1 first states that the equilibrium exists. Second, assuming that there is positive demand because at least one type makes a profit from investment, it claims that the equilibrium price is unique, and characterizes its relation with the borrowing constraint.[7] The constraint is binding at a price smaller than 1, but slack if $q_1 = 1$. At this price, investors are indifferent between purchasing or selling claims.

The two equivalences in part (*ii*) of Proposition 1 formalize this indifference property. They imply that either all or none of the investors are constrained by the borrowing limit. It is sufficient that only one group of investors is forced to liquidate claims on the market, i.e. $\tilde{l}_1^j > 0$, to reduce the price q_1 to a level below one. This deflation either constrains other investors via a tighter borrowing limit, or it gives them a pecuniary incentive to issue as much debt as possible. They do so to purchase additional claims, i.e. $\tilde{l}_1^j < 0$. To see this, recall the budget constraints (1) and (2), and note that, provided $q_1 < 1$, every purchased unit of claims offers a positive return $1 - q_1 > 0$ in the final period. Hence, in order to transfer funds to $t = 2$, solvent investors prefer additional investment $\tilde{l}_1^j < 0$ over savings $\tilde{d}_1^j < 0$. For a price $q_1 = 1$, they are indifferent between both ways of intertemporal substitution.[8]

3.3 Equilibrium effects of variations in beliefs

In this section, we analyze how variations in beliefs affect the allocation and prices in the competitive equilibrium. We show how the two main ingredients of our model, the financial friction and heterogeneity of beliefs, interact. The results of this comparative statics exercise allow us to specify how different types contribute to financial amplification, and how belief heterogeneity affects the overall probability of financial distress.

To keep the model tractable, we henceforth impose the following assumption without further mention.

Assumption 1. *Investors have exponential preferences of the form* $u\left(c_t^j\right) = -\exp\left(-Ac_t^j\right)$*, where absolute risk aversion* $A = -\frac{u''\left(c_t^j\right)}{u'\left(c_t^j\right)}$ *is constant (CARA).*[9]

The assumption that absolute risk aversion is constant is useful to simplify the comparative statics analysis below.

[7]However, the equilibrium price exists even if demand is zero, as this scenario corresponds to all investors being bankrupt, and infinitely many prices satisfy the Walrasian equilibrium definition. Abstracting from this case, we focus on equilibria with positive demand, which turn out to be uniquely determined.

[8]Formally, one of the Euler equations (4) and (5) is redundant in the unconstrained case, i.e. if $q_1 = 1$ and $\tilde{\eta}_1^j = 0$ for all j. Intuitively, investors are indifferent between the instruments $\tilde{l}_1^j$ and $\tilde{d}_1^j$, given that both promise a zero net return. We assume without loss of generality that there is no trade in the unconstrained economy, i.e. $\tilde{l}_1^j = 0$ for all j.

[9]For expositional reasons, we continue using the general notation $u\left(c_t^j\right)$.

We start out by examining the effect of changes in period-0 variables on the equilibrium price in $t = 1$, before analyzing how belief variations impact the equilibrium values of investment and borrowing in period $t = 0$. Note that the period-1 equilibrium price q_1 is no direct function of beliefs $\mathcal{F}$, but only through period-0 choices $\tilde{a}_0(\mathcal{F})$ and $\tilde{d}_0(\mathcal{F})$, i.e. $q_1 = q_1\left(\tilde{a}_0(\mathcal{F}), \tilde{d}_0(\mathcal{F})\right)$. Thus, this two-step procedure allows us to elaborate the relationship between the set of beliefs in the economy and the equilibrium price q_1, which defines the tightness of the borrowing constraint, and measures the extent of financial distress.

Period-0 allocation and the equilibrium price. Proposition 2 states how the equilibrium price q_1 is linked to period-0 levels of investment and debt.

Proposition 2.

(i) If investors hold heterogeneous beliefs $\mathcal{F}$, the period-1 equilibrium price q_1 is decreasing with period-0 investment and borrowing, i.e., for all j,

$$\frac{\partial q_1}{\partial \tilde{a}_0^j} < 0 \text{ and } \frac{\partial q_1}{\partial \tilde{d}_0^j} < 0.$$

(ii) If investors hold the homogeneous belief $F(R)$, the period-1 equilibrium price q_1 is decreasing with period-0 borrowing, i.e.

$$\frac{\partial q_1}{\partial \tilde{d}_0} < 0.$$

Proposition 2 states that more investment and borrowing in period $t = 0$ have a diminishing effect on the future equilibrium asset price. While the former is irrelevant in the homogeneous case, where trade does not occur, the negative effect of borrowing persists.

The two effects work through different channels, illustrated by the budget constraints (1) and (2). First, investment in $\tilde{a}_0^j$ increases period-2 consumption $\tilde{c}_2^j$ one-to-one, while $\tilde{c}_1^j$ rises with factor $\hat{R}$. Thus, in a sufficiently adverse state, satisfying $\hat{R} < 1$, consumption in the last period $\tilde{c}_2^j$ increases by more in response to investment than $\tilde{c}_1^j$. To smooth consumption, investors redistribute resources from $t = 2$ to $t = 1$ by liquidating $\tilde{l}_1^j$ units of their asset position (or purchasing less additional units). Second, higher indebtedness $\tilde{d}_0^j$ reduces the initial period-1 wealth $\hat{R}\tilde{a}_0^j - \tilde{d}_0^j$, raising the risk of being constrained and forced to liquidate a fraction of the portfolio. Both channels result in a higher supply (and a lower demand) of claims, which, in turn, reduce the equilibrium price q_1.

Beliefs and the period-0 allocation. We now turn to the relationship between investment $\tilde{a}_0(\mathcal{F})$ and borrowing $\tilde{d}_0(\mathcal{F})$ and investors' beliefs $\mathcal{F}$. To that end, we employ methods from the calculus of variation. We adopt the following procedure, that was first applied to heterogeneous belief environments by Dávila and Walther (2023). Recall that type-j investors' beliefs are characterized by the perceived distribution of R with cdf $F^j(R)$. Consider a perturbation to beliefs of the form $F^j(R)+\epsilon G^j(R)$, where $\epsilon>0$ is an arbitrary number, and $G^j(R)$ captures the direction of the perturbation. $F^j(R)+\epsilon G^j(R)$ is required to be a valid cdf for small enough ϵ, so we assume it is continuous and differentiable, satisfies $G\left(\underline{R}\right)=G\left(\overline{R}\right)=0$, and $\partial\left(F^j(R)+\epsilon G^j(R)\right)/\partial R\geq 0$ for sufficiently small ϵ.

This setup allows us to specify the concepts of optimism and pessimism. These terms are defined relative to each other in the sense of first-order stochastic dominance. A perturbation $G^j(R)$ makes type-j investors more optimistic if and only if it satisfies $F^j(R)+\epsilon G^j(R)\leq F^j(R)$ for all R. It is easy to see that a more optimistic belief requires the perturbation to have a non-positive direction, i.e. $G^j(R)\leq 0$ for all R. Analogously, investors of type j are made more pessimistic through a perturbation with direction $G^j(R)\geq 0$ for all R. Intuitively, investors are more optimistic if they assign lower probabilities than pessimists to low returns, so their cdf is shifted downwards.[10]

Using this technique, we show how a variation of a type's belief alters its individual choices of investment and debt issuance. The corresponding functional derivatives are

$$\frac{\delta\tilde{a}_0^j}{\delta F^j}\cdot G^j \text{ and } \frac{\delta\tilde{d}_0^j}{\delta F^j}\cdot G^j,$$

where δ denotes the operator for functional derivatives. Proposition 3 summarizes the results.

Proposition 3.

(i) *Let investors hold heterogeneous beliefs $\mathcal{F}$, and let $G^j(R)$ be the direction of a perturbation of type-j investors' belief $F^j(R)$. More optimistic (pessimistic) investors invest and borrow more (less), i.e.*

$$\frac{\delta\tilde{a}_0^j}{\delta F^j}\cdot G^j\begin{cases}\geq 0, & G^j(R)\leq 0\\ <0, & G^j(R)\geq 0\end{cases} \quad \textit{and} \quad \frac{\delta\tilde{d}_0^j}{\delta F^j}\cdot G^j\begin{cases}\geq 0, & G^j(R)\leq 0\\ <0, & G^j(R)\geq 0\end{cases}.$$

(ii) *Let investors hold the homogeneous belief $F(R)$, and let $G(R)$ be the direction of a perturbation. The more optimistic (pessimistic) the homogeneous belief is, the more*

[10] In the case of investors holding homogeneous beliefs, a perturbation implies a variation of the true distribution $F(R)$.

(less) investors borrow, i.e.

$$\frac{\delta \tilde{d}_0}{\delta F} \cdot G \begin{cases} \geq 0, & G(R) \leq 0 \\ < 0, & G(R) \geq 0 \end{cases}.$$

The essential insight from Proposition 3 is that investment and borrowing are monotone functions of beliefs. The more optimistic a group of investors is, the more it invests into the asset, and the more debt it issues. The opposite holds true for more pessimistic groups. If investors are homogeneous, only borrowing responds to variations in beliefs, while the asset is not traded.

Beliefs and the equilibrium price. Combining the results from Propositions 2 and 3, we describe how behavioral responses of investors to changes in beliefs $\mathcal{F}$ impact the period-1 equilibrium price $q_1\left(\tilde{a}_0\left(\mathcal{F}\right), \tilde{d}_0\left(\mathcal{F}\right)\right)$ in Theorem 1.

Theorem 1.

(i) Let investors hold heterogeneous beliefs $\mathcal{F}$.

(1) Let further $G^j(R)$ be the direction of a perturbation of type-j investors' belief $F^j(R)$, and beliefs $F^i(R)$ be constant for all $i \neq j$. If the perturbation makes investors of type j more optimistic (pessimistic), the period-1 equilibrium price q_1 is lower (higher), i.e.

$$\frac{\delta q_1}{\delta F^j} \cdot G^j \begin{cases} \leq 0, & G^j(R) \leq 0 \\ > 0, & G^j(R) \geq 0 \end{cases}.$$

(2) Let further $G^j(R) < 0 < G^i(R)$ with $|G^j(R)| = |G^i(R)|$ for all R be the directions of two perturbations that make investors of type j more optimistic, and investors of type i more pessimistic by the same magnitude. The behavioral responses to the perturbation with direction $G^j(R)$ have a stronger impact on the period-1 equilibrium price q_1 than those of the perturbation with direction $G^i(R)$, i.e.

$$\left|\frac{\delta q_1}{\delta F^j} \cdot G^j\right| \geq \left|\frac{\delta q_1}{\delta F^i} \cdot G^i\right|.$$

(ii) Let investors hold the homogeneous belief $F(R)$, and let $G(R)$ be the direction of a perturbation. If the perturbation makes investors more optimistic (pessimistic), the

period-1 equilibrium price q_1 is lower (higher), i.e.

$$\frac{\delta q_1}{\delta F} \cdot G \begin{cases} \leq 0, & G(R) \leq 0 \\ > 0, & G(R) \geq 0 \end{cases}.$$

Theorem 1 comprises the first set of key results of this paper. Part (i) characterizes the relationship of q_1 and heterogeneous beliefs. The more optimistic investors are, the lower the collateral price is in equilibrium. Conversely, if investors hold more pessimistic beliefs, the equilibrium price is higher. This result originates from the two monotonicities we have established in Propositions 2 and 3: q_1 responds monotonely to period-0 investment and borrowing, which, in turn, are monotonely driven by beliefs.

However, according to statement (2), the equilibrium price responds asymmetrically to symmetric variations of beliefs. Consider the thought experiment of two distinct perturbations, one making investors of type j more optimistic, the other making investors of type i more pessimistic, both to the very same extent. Formally, this is equivalent to decreasing type j's and increasing type i's probability mass for each realization $\hat{R}$ by the same amount. The statement argues that the perturbation to j dominates the perturbation to i. Thus, the equilibrium price turns out to be lower. More precisely, the perturbation to the optimistic type j exerts a downward effect that outweighs the upward effect from the perturbation to the pessimistic type i, resulting in a lower equilibrium price. The asymmetry between optimistic and pessimistic investors' influence on q_1 is the main result of Theorem 1, which we will use to derive optimal corrective policies in the following section.

Key to understand the asymmetry is the collateral constraint. By the two perturbations, type-j investors become more optimistic, willing to invest and borrow more, while type-i investors become more pessimistic, willing to invest less and save more. Importantly, both types have the incentive to invest into the asset as collateral in $t = 1$. In $t = 0$, this incentive amplifies type j's willingness to extend investment, but it counteracts type i's willingness to reduce investment. Accordingly, it induces type j to increase period-0 borrowing by more than type i increases period-0 savings. Therefore, when the constraint is binding in the following period $t = 1$, type-j investors' supply of liquidated claims will relatively exceed type-i investors' demand, which can only be equated for a lower equilibrium price q_1.

Part (ii) of Theorem 1 states that the former result holds true in the case of a homogeneous belief as well. A lower equilibrium price will arise if the uniform belief is more optimistic, and q_1 will be higher if it is more pessimistic.

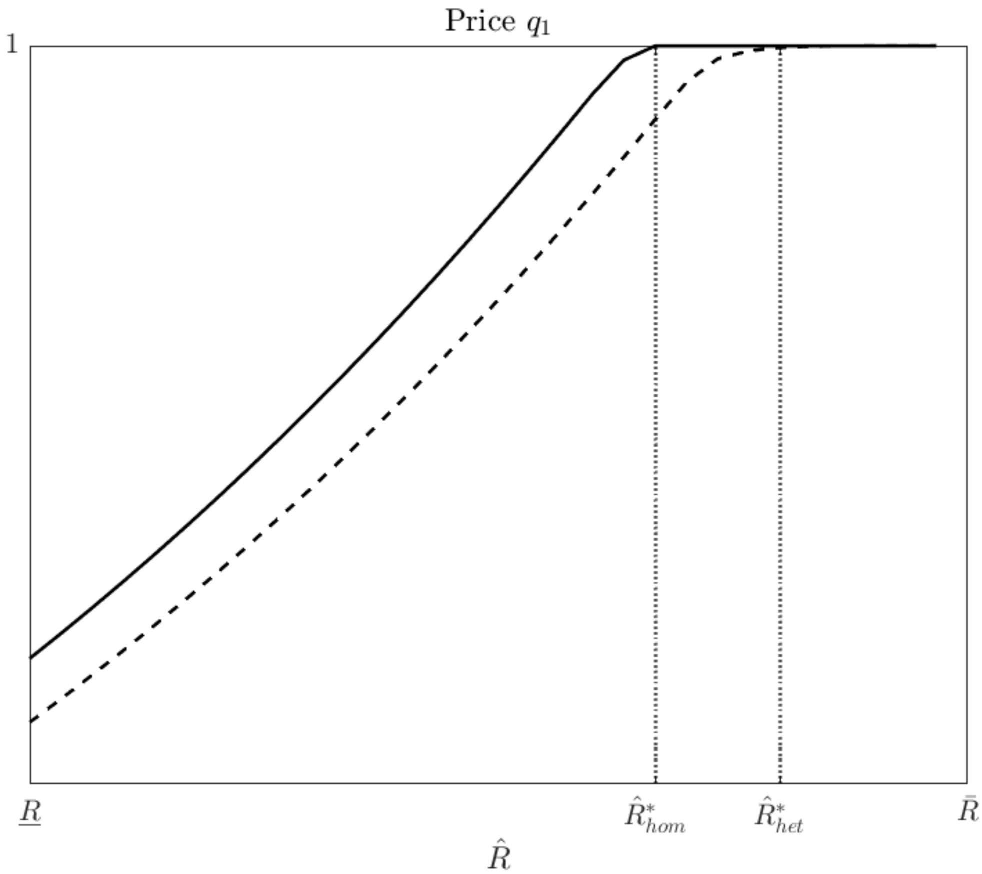

Figure 1: **Mapping from $\hat{R}$ to q_1**

Note: This figure shows the mapping from $\hat{R}$ to q_1 for the two cases when investors hold the homogeneous belief $F(R)$ or heterogeneous beliefs $\mathcal{F}$, respectively. The solid line refers to the homogeneous case, and the dashed line refers to the heterogeneous case. $\hat{R}^*_{hom}$ and $\hat{R}^*_{het}$ are thresholds as defined in Definition 3. The assumptions underlying this simulation are given in section 4.4.

Probability of financial distress. While Theorem 1 specifies how different types of investors contribute to financial amplification, we finally evaluate how heterogeneity affects the overall probability of financial distress. We apply the method proposed by Dávila and Walther (2023) to prove that financial distress is more likely under heterogeneous beliefs. The probability of financial distress is determined by the lowest possible realization of R such that the constraints are slack.

Definition 3. *Let* $\hat{R}^*_{het} \equiv \min\left\{\hat{R} \mid \tilde{\eta}^j_1 = 0 \textit{ for all } j\right\}$ *and* $\hat{R}^*_{hom} \equiv \min\left\{\hat{R} \mid \tilde{\eta}_1 = 0\right\}$ *be the lowest possible realizations of* R *such that the borrowing constraints are slack in the competitive equilibrium if investors hold heterogeneous beliefs* $\mathcal{F}$ *or the homogeneous belief* F*, respectively.*

Definition 3 translates into the mappings $\hat{R} \mapsto q_1\left(\hat{R}\right)$ as q_1 serves as a measure of financial distress, formally written as

$$q_1 \begin{cases} = 1 & \hat{R} \geq \hat{R}^*_{het} \\ < 1 & \hat{R} < \hat{R}^*_{het} \end{cases} \text{ or } q_1 \begin{cases} = 1 & \hat{R} \geq \hat{R}^*_{hom} \\ < 1 & \hat{R} < \hat{R}^*_{hom} \end{cases}.$$

in the heterogeneous and the homogeneous case, respectively. Figure 1 portrays an illustration of the two mappings.[11]

We show that the the threshold is lower if investors hold a homogeneous belief, compared to a setting of heterogeneous beliefs varying around it.

Theorem 2. *Consider two distinct populations with investors holding heterogeneous beliefs $\mathcal{F}$ in one, and the homogeneous belief $F(R)$ in the other. If the homogeneous belief is not more optimistic than any other belief in the heterogeneous case, i.e. $F^j(R) < F(R)$ for all R and at least one j, the probability of financial distress in the competitive equilibrium is higher under heterogeneity than under homogeneity, which is equivalent to*

$$\hat{R}^*_{het} > \hat{R}^*_{hom}.$$

Theorem 2 constitutes the second key result of our analysis. In an environment of heterogeneous beliefs, it is more likely that financial distress occurs. In general, it occurs whenever the realized return $\hat{R}$ is insufficient so that each investor could comply with her repayment obligations. If investors share a homogeneous belief, each $\hat{R} < \hat{R}^*_{hom}$ will constrain *all* investors. However, if beliefs are heterogeneous, it is enough that $\hat{R}$ is too low for *one* group to make everyone's borrowing constraint binding. In fact, under heterogeneity, the threshold $\hat{R}^*_{het}$ corresponds to the most optimistic type reaching the constraint, as it has built up the highest exposure to low returns.

We find that the most optimistic type − and all other types with it − is financially distressed even for higher returns compared to if they held a homogeneous belief. Consequently, under heterogeneity, financial distress occurs in even more favorable states of the world (as depicted in Figure 1), and is hence more likely. It rests on the presumption that the most optimistic belief is sufficiently off the ex post realization.

Hence, Theorem 2 highlights that financial distress may have an additional source. As is well known from the literature, a spiral of financial amplification can be initiated by adverse shocks sufficiently strong to drive excessively borrowing agents towards the constraint. Beyond that, we document that the dispersion of beliefs lays the ground for another source of distress, namely that some agents' beliefs deviate sufficiently from the true shock distribution.

[11] Figure 1 is based on the numerical application provided in section 4.4.

3.4 Discussion

In the previous section, we have shown that belief heterogeneity increases the probability of financial distress, and how it affects the equilibrium collateral price. This price, in turn, is the main determinant of the financial friction, as it governs the tightness or slackness of the borrowing limit. Theorems 1 and 2 thus allow us to characterize the interaction of the collateral constraint and belief divergence, and to specify how different types of agents contribute to financial amplification. The mechanism emerging from this interaction has two features.

The first property is that heterogeneity of beliefs raises the likelihood of financial distress relative to the homogeneous benchmark. If investors have diverging expectations of future returns ex ante, some of these will differ from ex post realizations, which is sufficient to constrain all investors' borrowing. In contrast, under the homogeneous benchmark, when investors have rational expectations, the constraint binds only if the ex post realization is starkly adverse for all. Therefore, we conclude that belief disagreements facilitate the triggering of financial amplification.

The second feature refers to different investors' contributions to the financial amplification mechanism. Principally, during financial distress, optimistic and pessimistic investors drive collateral prices in opposing directions, as the former tend to sell, and the latter tend to purchase. However, we find an asymmetry of their contributions, attributing a larger impact to optimistic behavior. Hence, to distinguish the behavior of borrowing constraints in the presence of heterogeneous beliefs from the homogeneous benchmark, we must take into account how beliefs are distributed over the population.

We find that financial frictions are more severe under heterogeneity rather than homogeneity if the mean belief coincides with, or is more optimistic than the homogeneous belief. This implies that, so long as the belief distribution is symmetric around the homogeneous belief, or skewed towards more optimistic beliefs, heterogeneity exacerbates the financial amplification mechanism. The reason is that optimistic investors' (negative) contribution more than outweighs pessimistic investors' (positive) contribution.[12]

Our approach to financial amplification goes beyond the existing literature. These studies, presuming rational expectations, establish mechanisms where financial constraints

[12] Belief heterogeneity may mitigate financial amplification compared to the homogeneous benchmark, on the contrary, provided that the distribution is sufficiently skewed towards more pessimistic beliefs. The skewness would have to be large enough to reverse the relation of optimistic and pessimistic investors' influence on the collateral price. However, we argue that the presumption of a symmetric distribution is likely to prevail in financial markets. A range of studies provides both empirical and theoretical evidence that financial market participants' beliefs are distributed symmetrically, if not (close to) normally (Söderlind, 2009; Cvitanic and Malamud, 2011; Atmaz, 2014; Atmaz and Basak, 2016). Under this premise, extreme beliefs are either sufficiently improbable or counteracted by an equiprobable set of contrasting beliefs.

bind in response to exogenous reductions of aggregate investment or aggregate net worth (Bianchi, 2011; Dávila and Korinek, 2018; Jeanne and Korinek, 2020). We extend this approach, and show that belief differences are sufficient to make such constraints binding. We may further quantify how market participants contribute to their tightness on the micro level.

In the following section, we turn to the welfare implications of the interaction mechanism between heterogeneous beliefs and financial frictions, which we have established hitherto.

4 Efficiency analysis

We proceed by exploring the efficiency properties of our baseline economy. Given that the borrowing constraint is price-dependent, investors are subject to a pecuniary externality, as they do not internalize how their decisions affect other agents' individual welfare. We characterize these uninternalized welfare effects and their interplay with belief heterogeneity in the following section. Subsequently, we derive the constrained-efficient allocation as a welfare benchmark to contrast the competitive equilibrium, and develop optimal corrective policies that allow to implement it. Lastly, we quantify the welfare impact of such policy interventions numerically.

4.1 Uninternalized welfare effects

The collateral price q_1 links individual choices and utilities across investors in two ways. First, it changes the value of investors' budgets in $t = 1$. Second, it determines the tightness of the borrowing constraints. Investors do not internalize these price effects. We use the terminology of Dávila and Korinek (2018) of *distributive* and *collateral* externalities.

Definition 4. *The uninternalized effects of changes in any type j's aggregate state variables $\left\{\tilde{a}_0^j, \tilde{d}_0^j\right\}$ on any i's individual welfare in periods $t = 1, 2$ can be written as*

$$\frac{\partial V_1^i}{\partial \tilde{a}_0^j} = \tilde{\lambda}_1^i D^i_{\tilde{a}_0^j} + \tilde{\eta}_1^i C^i_{\tilde{a}_0^j}$$
$$\frac{\partial V_1^i}{\partial \tilde{d}_0^j} = \tilde{\lambda}_1^i D^i_{\tilde{d}_0^j} + \eta_1^i C^i_{\tilde{d}_0^j}.$$

where $D^i_{\tilde{a}_0^j}$ and $D^i_{\tilde{d}_0^j}$ are referred to as distributive externalities, and $C^i_{\tilde{a}_0^j}$ and $C^i_{\tilde{d}_0^j}$ are referred to as collateral externalities.

(i) If investors hold heterogeneous beliefs $\mathcal{F}$, distributive externalities are given by

$$D^i_{\tilde{a}^j_0} = \frac{\partial q_1}{\partial \tilde{a}^j_0}\tilde{l}^i_1,$$
$$D^i_{\tilde{d}^j_0} = \frac{\partial q_1}{\partial \tilde{d}^j_0}\tilde{l}^i_1,$$

and collateral externalities are given by

$$C^i_{\tilde{a}^j_0} = \phi\frac{\partial q_1}{\partial \tilde{a}^j_0}\left(\tilde{a}^i_0 - \tilde{l}^i_1\right),$$
$$C^i_{\tilde{d}^j_0} = \phi\frac{\partial q_1}{\partial \tilde{d}^j_0}\left(\tilde{a}^i_0 - \tilde{l}^i_1\right).$$

(ii) If investors hold the homogeneous belief $F(R)$, distributive externalities are zero, and collateral externalities are given by

$$C_{\tilde{a}_0} = \phi\frac{\partial q_1}{\partial \tilde{a}_0}\bar{a},$$
$$C_{\tilde{d}_0} = \phi\frac{\partial q_1}{\partial \tilde{d}_0}\bar{a}.$$

Distributive effects describe the price-induced redistribution between trading agents, altering their marginal rates of substitution. Collateral effects measure the price-induced change in an agent's capacity to borrow.

In an environment of heterogeneous beliefs, it turns out that, the more optimistic investors are, the more likely it is that they will sell claims on the asset in $t = 1$ ($\tilde{l}^j_1 \geq 0$). Accordingly, more pessimistic investors will more probably enter the market as buyers ($\tilde{l}^j_1 < 0$). The reason is that a group's exposure to adverse states, reflected by its position $\tilde{a}^j_0$, is a monotone function of beliefs (see Proposition 3). We use this fact, as well as Proposition 2, to characterize the direction of distributive and collateral externalities.

Proposition 4.

(i) If investors hold heterogeneous beliefs $\mathcal{F}$, distributive externalities have a non-positive sign for period-1-sellers, i.e. $D^i_{\tilde{a}^j_0} \leq 0$ and $D^i_{\tilde{d}^j_0} \leq 0$ if $\tilde{l}^i_1 \geq 0$, and a non-negative sign for period-1-buyers, i.e. $D^i_{\tilde{a}^j_0} \geq 0$ and $D^i_{\tilde{d}^j_0} \geq 0$ if $\tilde{l}^i_1 \leq 0$. If investors hold the homogeneous belief $F(R)$, distributive externalities are zero.

(ii) Collateral externalities have a non-positive sign for any type i of investors, and irrespective of beliefs, i.e. $C^i_{\tilde{a}^j_0} \leq 0$ and $C^i_{\tilde{d}^j_0} \leq 0$ for each i.

Distributive externalities are signed reflective on the fact that a decline of the equilibrium price q_1 benefits buyers and harms sellers in $t = 1$. Collateral externalities, in turn, are unambiguously adverse to each type of agent, as more investment and borrowing reduce the collateral value, cutting any investor's borrowing capacity.

Ultimately, we evaluate the welfare implications of the interaction mechanism between beliefs and the equilibrium price q_1, which we have established in Theorem 1.

Proposition 5.

(i) Let investors hold heterogeneous beliefs $\mathcal{F}$.

(1) Let further $G^j(R)$ be the direction of a perturbation of type-j investors' belief $F^j(R)$, and beliefs $F^i(R)$ be constant for all $i \neq j$. If the perturbation makes investors of type j more optimistic (pessimistic), both distributive and collateral externalities of any type-i investor are larger (smaller) in absolute value, i.e., for each $i \neq j$ and $x \in \{a, d\}$,

$$\left| \frac{\delta D^i_{\tilde{x}^j_0}}{\delta F^j} \cdot G^j \right| \begin{cases} \geq 0, & G^j(R) \leq 0 \\ \leq 0, & G^j(R) \geq 0 \end{cases} \quad \textit{and} \quad \frac{\delta C^i_{\tilde{x}^j_0}}{\delta F^j} \cdot G^j \begin{cases} \leq 0, & G^j(R) \leq 0 \\ \geq 0, & G^j(R) \geq 0 \end{cases}.$$

(2) Let further $G^j(R) < 0 < G^k(R)$ with $|G^j(R)| = \left|G^k(R)\right|$ for all R be the directions of two perturbations that make investors of type j more optimistic, and investors of type k more pessimistic by the same magnitude. The uninternalized welfare effects under the perturbation with direction $G^j(R)$ are stronger than those under the perturbation with direction $G^k(R)$, i.e., for each $i \neq j, k$ and $x \in \{a, d\}$,

$$\left| \frac{\delta D^i_{\tilde{x}^j_0}}{\delta F^j} \cdot G^j \right| \geq \left| \frac{\delta D^i_{\tilde{x}^k_0}}{\delta F^k} \cdot G^k \right| \quad \textit{and} \quad \left| \frac{\delta C^i_{\tilde{x}^j_0}}{\delta F^j} \cdot G^j \right| \geq \left| \frac{\delta C^i_{\tilde{x}^k_0}}{\delta F^k} \cdot G^k \right|.$$

(ii) Let investors hold the homogeneous belief $F(R)$, and let $G(R)$ be the direction of a perturbation. If the perturbation makes investors more optimistic (pessimistic), collateral externalities are larger (smaller) in absolute value, i.e., for $x \in \{a, d\}$

$$\frac{\delta C_{\tilde{x}_0}}{\delta F} \cdot G \begin{cases} \leq 0, & G(R) \leq 0 \\ \geq 0, & G(R) \geq 0 \end{cases}.$$

Proposition 5 describes the welfare effects associated with the interaction of beliefs and the equilibrium price q_1. It states that more optimistic types exerting downward pressure on the collateral price, due to large investment and borrowing, impose more intense negative distributive externalities on sellers ($\tilde{l}_1^i > 0$), and more intense positive ones on buyers ($\tilde{l}_1^i < 0$). In contrast, more pessimistic types' choices have an increasing impact on the collateral price, by this causing the reverse response of distributive externalities.

By the same logic, collateral externalities, being non-positive in general, turn out to be more or less pronounced in the case of more optimistic or pessimistic groups, respectively. This result holds true analogously in the homogeneous case.

Importantly, the asymmetry between optimistic and pessimistic investors' influence on q_1 translates into asymmetric welfare effects, as we formalize in statement (2) of part (i). Since the price responds more markedly to optimistic than pessimistic behavior, the former further dominates in welfare terms. If the two groups j's and k's beliefs are made more optimistic and pessimistic to the same extent, respectively, any further type i's group-wide welfare losses from j's high investment and borrowing exceed the gains from k's precaution.

4.2 Constrained efficiency

Investors do not internalize the distributive or collateral side effects of their behavior which materialize through the collateral price q_1. These externalities render the competitive equilibrium allocation inefficient. To evaluate its welfare properties, we employ the concept of constrained efficiency.

The constrained-efficient allocation solves the problem of a constrained social planner who chooses investment and borrowing in period $t = 0$, while leaving all later choices to private agents. Specifically, she maximizes social welfare subject to all resource constraints, technological constraints, market clearing conditions, and financial frictions, respecting the competitive equilibrium price formation (see equation (6)).

Social welfare is evaluated by aggregating investors' expected lifetime utilities, and applying arbitrary Pareto weights $\omega = \{\omega^j\}_{j\in\{1,\dots,J\}}$. A relevant question in this setting is the planner's belief (Blume et al., 2018; Dávila, 2023; Kim and Kim, 2021). If we assigned a specific belief to the planner, she would naturally disagree with investors upon their beliefs. Abstracting from this trivial motive of correction, we aim at isolating ex ante corrective policies related to the financial friction, and, thus, make the following assumption.

Assumption 2. *The constrained social planner has no superior information, and respects individual beliefs for each type j.*

We solve the following social planner problem.

$$\max_{\left\{\tilde{c}_0^j,\tilde{a}_0^j,\tilde{d}_0^j\right\}_{j\in\{1,\ldots J\}}} \sum_{j=1}^{J} \omega^j s^j \left[u\left(\tilde{c}_0^j\right) + E^j\left[V_1^j\left(\tilde{a}_0^j, \tilde{d}_0^j|\tilde{a}_0, \tilde{d}_0\right)\right]\right] \quad \text{s.t.}$$

$$(\tilde{\lambda}_0) \quad \sum_{j=1}^{J} s^j \tilde{c}_0^j = \sum_{j=1}^{J} s^j \left[e + \tilde{d}_0^j\right] \tag{12}$$

$$(\tilde{\psi}) \quad \sum_{j=1}^{J} s^j \tilde{a}_0^j = \bar{a}.$$

With the first order conditions for consumption, $\tilde{\lambda}_0 = \omega^j u'\left(\tilde{c}_0^j\right)$, the planner's optimality conditions for each j are

$$0 = E^j\left[Ru'\left(\tilde{c}_1^j\right) + u'\left(\tilde{c}_2^j\right) + \tilde{\eta}_1^j \phi q_1\right] - \frac{\tilde{\psi}}{\omega^j} + \sum_{i=1}^{J} \frac{\omega^i}{\omega^j}\frac{s^i}{s^j} E^i\left[D^i_{\tilde{a}_0^j} u'\left(\tilde{c}_1^i\right) + \tilde{\eta}_1^i C^i_{\tilde{a}_0^j}\right] \tag{13}$$

$$0 = u'\left(\tilde{c}_0^j\right) - E^j\left[u'\left(\tilde{c}_1^j\right)\right] + \sum_{i=1}^{J} \frac{\omega^i}{\omega^j}\frac{s^i}{s^j} E^i\left[D^i_{\tilde{d}_0^j} u'\left(\tilde{c}_1^i\right) + \tilde{\eta}_1^i C^i_{\tilde{d}_0^j}\right]. \tag{14}$$

We can now define the constrained-efficient allocation.

Definition 5. *The period-0 allocation* $\left\{\tilde{c}_0^j, \tilde{a}_0^j, \tilde{d}_0^j\right\}_{j\in\{1,\ldots J\}}$ *is constrained-efficient if and only if there are shadow prices* $\tilde{\lambda}_0$, $\tilde{\psi}$, $\left\{\tilde{\eta}_1^j\right\}_{j\in\{1,\ldots J\}}$, *and a set of Pareto weights* $\left\{\omega^j\right\}_{j\in\{1,\ldots J\}}$ *such that it satisfies the price relation* (6) *for each* j, *the market clearing condition* (10), *and the resource constraint* (12), *as well as the equations* (13), (14), *and* $\tilde{\lambda}_0 = \omega^j u'\left(\tilde{c}_0^j\right)$ *for each* j, *given population shares* s *and beliefs* $\mathcal{F}$.

Equations (13) and (14) differ from the competitive equilibrium conditions (8) and (9) through the aggregate terms of externalities on the right-hand side. They indicate formally that the competitive allocation is not constrained-efficient, whereas the social planner takes distributive and collateral externalities into account. Furthermore, she accounts for market clearing in $t = 0$, represented by the multiplier $\tilde{\psi}$.

4.3 Optimal corrective policies

The constrained-efficient allocation can be achieved in a decentralized market using a set of adequate policy instruments. We start out by characterizing optimal corrective taxes under both heterogeneous and homogeneous beliefs. We contrast a system of non-linear taxes under heterogeneity with a simple linear Pigouvian tax. The latter allows us to quantify welfare differences between our approach and previous policy proposals in the following section.

Decentralization. To decentralize the constrained-efficient allocation, we provide the social planner with access to Pigouvian taxes, available to manipulate agents' investment and borrowing decisions, and lump-sup transfers. These instruments satisfy the conditions stated in the following proposition.

Proposition 6.

(i) If investors hold heterogeneous beliefs $\mathcal{F}$, the social planner can implement the constrained-efficient allocation by taxing investment and borrowing, satisfying

$$\tau_a^j = \operatorname{sgn}\left(\bar{a} - \tilde{a}_0^j\right)\left(s^j q_0 \tilde{\lambda}_0\right)^{-1} \sum_{i=1}^{J} \omega^i s^i E^i \left[D^i_{\tilde{a}_0^j} u'\left(\tilde{c}_1^i\right) + \tilde{\eta}_1^i C^i_{\tilde{a}_0^j}\right] \tag{15}$$

$$\tau_d^j = -\operatorname{sgn}\left(\tilde{d}_0^j\right)\left(s^j \tilde{\lambda}_0\right)^{-1} \sum_{i=1}^{J} \omega^i s^i E^i \left[D^i_{\tilde{d}_0^j} u'\left(\tilde{c}_1^i\right) + \tilde{\eta}_1^i C^i_{\tilde{d}_0^j}\right] \tag{16}$$

for each j, and rebating revenues through type-specific lump-sum transfers $T^j = \tau_a^j \operatorname{sgn}\left(\bar{a} - \tilde{a}_0^j\right) q_0 \left(\bar{a} - \tilde{a}_0^j\right) + \tau_d^j \operatorname{sgn}\left(\tilde{d}_0^j\right) \tilde{d}_0^j$.[13]

(ii) If investors hold the homogeneous belief $F(R)$, the social planner can implement the constrained-efficient allocation by taxing borrowing, satisfying

$$\tau_d = -\tilde{\lambda}_0^{-1} E\left[\tilde{\eta}_1 C_{\tilde{d}_0}\right], \tag{17}$$

and rebating revenues through lump-sum transfers $T = \tau_d \tilde{d}_0$, while the tax on investment is arbitrary.

In the heterogeneous case, our optimal Pigouvian taxes are characterized by a range of sufficient statistics related to distributive and collateral externalities, aggregated in the squared brackets in equations (15) and (16).[14]

Three components determine distributive effects. First, when price movements induce a redistribution of funds between period-1-buyers and -sellers, this affects their marginal rates of substitution. Second, price movements themselves measure the intensity of redistribution. Third, the direction of redistribution depends on whether an investor is a seller ($\tilde{l}_1^j > 0$) or a buyer ($\tilde{l}_1^j < 0$) in $t = 1$. The latter two components are captured by the distributive externalities $D^i_{\tilde{a}_0^j}$ and $D^i_{\tilde{d}_0^j}$, given in Definition 4.

Collateral effects are driven by another three components. First, the multiplier $\tilde{\eta}_1^j$ measures the welfare gain (loss) when the constraint is relaxed (tightened) by one unit. Second, price movements describe the change in an investor's borrowing capacity per unit

[13] We use a sign operator for an easier interpretation of taxes and subsidies, given the fact that investors can take short and long positions in the asset, as well as borrow and save.

[14] For a more detailed description of sufficient statistics, see Dávila and Korinek (2018).

of collateral, whose total magnitude available matters third. The last two elements are incorporated in the collateral externalities $C^i_{\tilde{a}^j_0}$ and $C^i_{\tilde{d}^j_0}$ from Definition 4.

If, however, investors hold the homogeneous and rational belief, these sufficient statistics turn out to be vastly simplified. Since investors do not trade the asset under homogeneity, the social planner cannot manipulate investment decisions. The resulting tax on investment is arbitrary. Moreover, for the very same reason, distributive externalities are zero, rendering the tax on borrowing responsive solely to collateral externalities (see equation (17)).

Notably, the instruments derived in Proposition 6 may well be subsidies instead of taxes, depending on the extent of externalities induced by type j, and its specific choices of investment and borrowing. Taxes/subsidies turn out to be zero only if *all* investors expect their collateral constraints to be slack. To put it another way, it suffices that one group of investors expects to be constrained to let taxes/subsidies take on either sign for the entire population. We will return to the signing of policy instruments in the next section.

Incentive compatibility. In an environment of heterogeneous agents, whose type is their private information, corrective policies may not be incentive-compatible. The instruments we have derived in Proposition 6 are type-specific, raising the question of knowledge required by the social planner to impose taxes in an incentive-compatible way.

Importantly, the optimal non-linear taxes in equations (15) and (16) incorporate no more than publicly known objects. To be precise, to set group-specific taxes, the social planner must be informed about the set of beliefs $\mathcal{F}$ in the economy, each type's respective population share s^j, as well as investment and borrowing choices $\tilde{a}_0$ and $\tilde{d}_0$, which are publicly observable in the market. Since the latter are monotone functions of beliefs, as we have shown in Proposition 3, they perfectly reveal any investor's belief.

Therefore, the constrained-efficient allocation can be implemented by means of the following system of non-linear Pigouvian taxes.

Theorem 3. *If investors hold heterogeneous beliefs $\mathcal{F}$, the social planner can implement the constrained-efficient allocation by taxing investment and borrowing according to the tax system $(\tilde{\tau}_a, \tilde{\tau}_d)$, satisfying*

$$\tilde{\tau}_a: \quad \tilde{a}^k_0 \mapsto \tilde{\tau}_a(\tilde{a}^k_0) \ s.t. \ \tilde{\tau}_a(\tilde{a}^k_0) = \begin{cases} RHS \ of \ (15) & if \ \tilde{a}^k_0 = \tilde{a}^j_0 \ for \ any \ j \ with \ \tilde{a}^j_0 \in \tilde{a}_0 \\ \infty & if \ \tilde{a}^k_0 \notin \tilde{a}_0 \end{cases} \tag{18}$$

$$\tilde{\tau}_d: \quad \tilde{d}^k_0 \mapsto \tilde{\tau}_d(\tilde{d}^k_0) \ s.t. \ \tilde{\tau}_d(\tilde{d}^k_0) = \begin{cases} RHS \ of \ (16) & if \ \tilde{d}^k_0 = \tilde{d}^j_0 \ for \ any \ j \ with \ \tilde{d}^j_0 \in \tilde{d}_0 \\ \infty & if \ \tilde{d}^k_0 \notin \tilde{d}_0, \end{cases} \tag{19}$$

and corresponding lump-sum transfers.

The essential point of Theorem 3 is that the social planner does not rely on knowledge of individual beliefs. The peculiar nature of our optimal Pigouvian taxes ensures that the constrained-efficient allocation is indeed decentralizable, even in a setting of heterogeneous beliefs.

Our results on optimal corrective policies give rise to several issues linked to the welfare implications of the interplay between belief heterogeneity and the financial friction. First, analyzing the responses of group-specific taxes/subsidies to variations of beliefs is informative on different types' contributions to changes in social welfare. Second, we seek to compare the efficiency properties of our economy under homogeneity and heterogeneity of beliefs. Third, it is enlightening to evaluate how the probability of financial distress is altered through a planner intervention of the kind sketched above.

Moreover, we aim at quantifying the welfare impact of the non-linear tax instruments we propose in contrast to a linear Pigouvian tax on borrowing. The latter is a standard macroprudential instrument which has gained much attention in the literature (Bianchi, 2011; Dávila and Korinek, 2018; Jeanne and Korinek, 2019, 2020). In our model, it corresponds to equation (17), being a tax on borrowing calibrated to the case of homogeneous and rational expectations.

Examining these questions is analytically intractable. The clear signing of tax instruments depends on the specific belief distribution, which we have kept general thus far. To gain insights into the welfare implications of our policy proposals, we provide a numerical application of our model in the following.

4.4 Numerical application

The numerical analysis requires a simplified version of our model. In this section, we first describe the simplifications applied to make the baseline model numerically tractable, and briefly characterize the resulting equilibrium allocations, prices, and, importantly, optimal corrective policies for different levels of belief heterogeneity. Subsequently, we quantify the welfare implications of such policies, and assess how these interventions impact the probability of financial distress.

Simplifications. Suppose the economy is populated by two groups of investors, called optimists and pessimists, indexed by o and p. We let both groups be of equal mass, i.e. $s^o = s^p = 1$, and differ in terms of their return expectations, i.e. $E^o[R] > E^p[R]$. Furthermore, there are only two states of the world. To be precise, R may take on either a *good* or a *bad* value, denoted by $R^g > R^b$.

Parameter		Value
Margin requirement	ϕ	0.35
Good state	R^g	2
Bad state	R^b	0
Initial endowment of consumption goods	e	1
Initial asset endowment	$\bar{a}$	2
Risk aversion	A	0.5
Heterogeneity step	μ	0.025
Initial belief	π^g	0.5

Table 1: **Parameter values**

We choose parameters in line with the assumptions underlying our theoretical analysis, simulating equilibria with significant trade volumes and binding financial constraints. Table 1 summarizes the parameter values chosen in the application.

The parameter ϕ, capturing the margin requirement for borrowing, is selected following Mendoza (2002) and Bianchi (2011), who suggest that debt is required to not exceed a fraction of 30 to 40 percent of tradable assets. Averaging these values, we set $\phi = 0.35$. The two states R^g and R^b are chosen with the aim to make trading incentives strong enough, which, in turn, ensures a significant trade volume. This condition is met for $R^g = 2$ and $R^b = 0$. For the same argument, we set initial endowments of consumption goods e and assets $\bar{a}$ to $e = 1$ and $\bar{a} = 2$, and choose a moderate degress of risk aversion $A = 0.5$.

Heterogeneity itself is defined as the linear distance between the probabilities that the two types assign to the good state, i.e. $\pi^{j,g} = 1 - \pi^{j,b}$. We increase this distance symmetrically by N steps of size $\mu = 0.025$ (see Simsek (2013) for comparison). The multiples N thus serve as a measure of belief heterogeneity. The benchmark case is a population with homogeneous beliefs, where $\pi^{o,g} = \pi^{p,g} \equiv \pi^g$, which we set to $\pi^g = 0.5$. Finally, the two types' beliefs at any given level of heterogeneity N are given by

$$E^o[R] = (\pi^g + N\mu)R^g + (\pi^b - N\mu)R^b$$
$$E^p[R] = (\pi^g - N\mu)R^g + (\pi^b + N\mu)R^b.$$

Notably, we let the social planner apply Pareto weights ω such that the constrained-efficient allocation replicates the unconstrained competitive allocation, i.e. when the collateral constraints are slack. This choice ensures that the simulated corrective interventions by the planner are solely related to inefficiencies from the financial friction, but not to differences in the aggregation of social welfare.

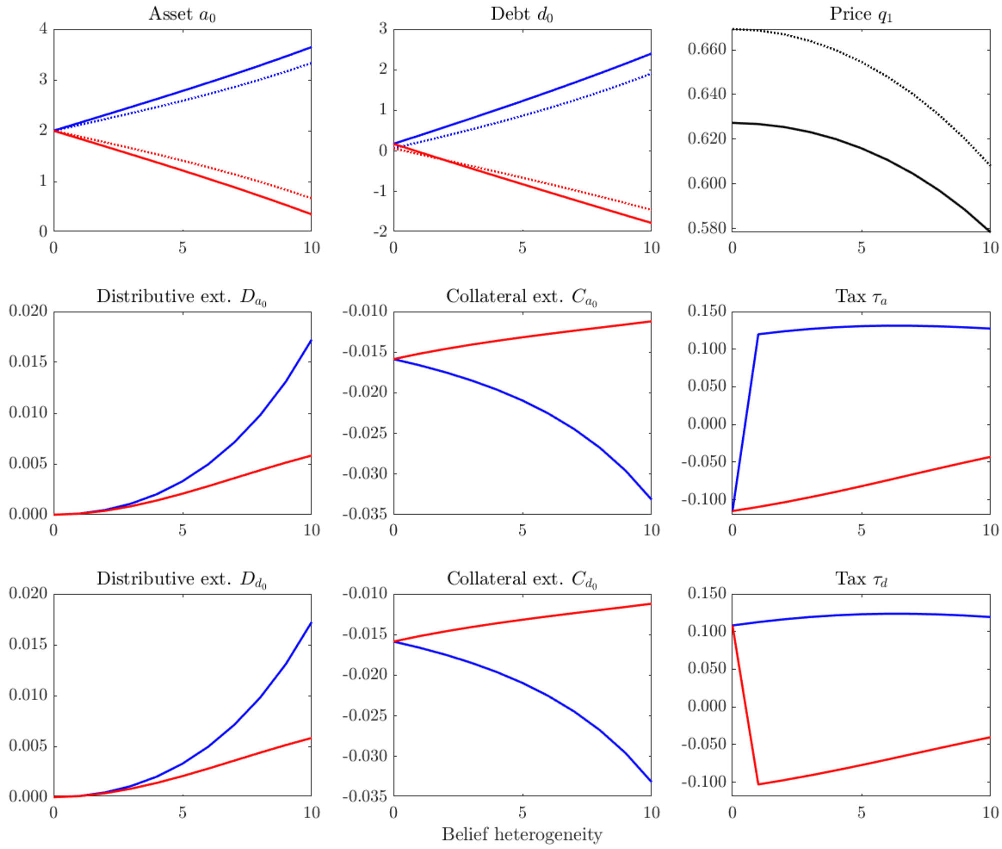

Figure 2: **Equilibrium allocations, prices, and optimal corrective policies**

Note: The three upper panels show period-0 choices of investment, and borrowing, as well as the period-1 asset price. The three middle panels show optimal taxes on investment, and aggregate distributive and collateral externalities therein. The three middle panels show optimal taxes on borrowing, and aggregate distributive and collateral externalities therein. The blue and red lines refer to the optimistic and the pessimistic type, respectively. Solid lines refer to variables from the competitive equilibrium, while dotted lines refer to the constrained-efficient equilibrium. Each number on the x-axis relates to the N-th heterogeneity step, where $N = 0$ stands for the benchmark case of homogeneous beliefs.

Allocations, prices, and corrective policies. Figure 2 displays the responses of key variables to different levels of heterogeneity. Specifically, it shows the equilibrium values of period-0 investment and borrowing, the period-1 price q_1 – the main determinant of the collateral constraint – as well as taxes and the externalities therein. The two beliefs diverge increasingly the further one follows the x-axis. The blue and red lines refer to the optimists and pessimists, respectively. Solid lines refer to variables from the competitive equilibrium, while dotted lines refer to the constrained-efficient equilibrium.

The top-left and top-central panels illustrate the monotonicity of period-0 investment and borrowing in beliefs. Starting from a no-trade equilibrium under homogeneous beliefs, where investors keep their initial asset position constant, investment and borrowing increase (decrease) the more optimistic (pessimistic) they become. Contrasting the competitive allocation, the social planner induces agents to trade, borrow, and save less. Importantly, the planner reduces optimists' borrowing by more than pessimists' saving, reflecting the asymmetry between optimistic and pessimistic types' contributions to financial distress, formalized in Theorem 1.

In the top-right panel, this asymmetry becomes evident in the response of the equilibrium price q_1 to increasing belief heterogeneity. Given that the influence of optimistic behavior is dominant, the equilibrium price declines even though we have not altered the economy's mean belief, but made the two types more heterogeneous in a symmetric manner. Furthermore, the social planner improves on the competitive allocation by sustaining a higher price, alleviating the tightness of the financial friction.

The panels in the second row of Figure 2 depict the aggregate distributive and collateral externalities associated with each type's investment, and the corresponding corrective policies, formalized in equation (15). To achieve constrained efficiency, the planner taxes investment by optimists ($\tau_a^o > 0$), and subsidizes asset purchases by pessimists ($\tau_a^p < 0$).

The interplay of aggregate distributive and collateral externalities determine the signs of the instruments. The tax on optimists' investment is driven by negative collateral externalities clearly outweighing positive distributive externalities. The latter arise because pessimists, buying claims in $t = 1$, benefit from the price decline induced by optimists' behavior. However, as the collateral price continues falling with increasing heterogeneity, optimists pass over more intense collateral externalities to pessimists.

Pessimists, in contrast, are subsidized because their cautious investment decisions tend to mitigate the price decline, benefiting optimists' budget in $t = 1$, and reducing collateral externalities. Since they behave with more precaution the more pessimistic they become, the social planner is less inclined to correct their behavior, and the subsidy reverts to zero.

The lower panels of Figure 2 refer to aggregate externalities associated with borrowing and saving, and the respective policy instruments, captured by equation (16). By the same mechanisms as for the correction of investment, borrowing by optimists is increasingly taxed ($\tau_d^o > 0$), and borrowing by pessimists is subsidized ($\tau_d^p < 0$).[15] If the two types of investors hold the homogeneous beliefs, their borrowing is slightly taxed.

Welfare effects. Thus far, we have qualified both the direction and the extent of corrective taxes. In the following, we turn to the normative question of how the Pigouvian correction translates into social welfare. We are particularly interested in measuring wel-

[15] Aggregate distributive and collateral externalities from borrowing turn out to be equal to those from investment in this example due to our assumption $R^b = 0$. In this case, price effects are identical, and so are type-specific externalities (see Definition 4).

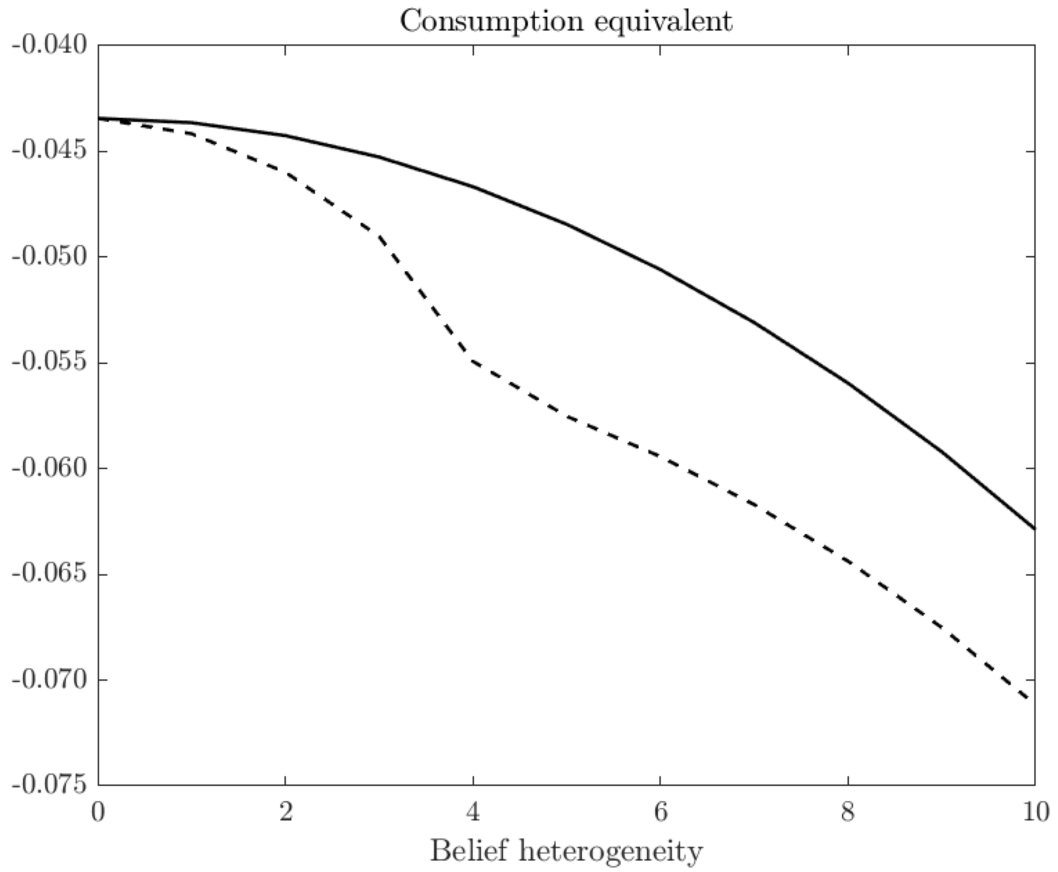

Figure 3: **Welfare effects of linear and non-linear corrective taxes**

Note: This figure shows the consumption equivalents of two types of allocations relative to the unconstrained competitive allocation. The solid line refers to constrained-efficient allocations, which are implemented by means of the system of non-linear taxes proposed in Theorem 3. The dotted line refers to allocations implemented by means of the system of linear taxes proposed in part (ii) of Proposition 6. Each number on the x-axis relates to the N-th heterogeneity step, where $N = 0$ stands for the benchmark case of homogeneous beliefs.

fare gains from the non-linear tax policy as opposed to a linear Pigouvian tax system, which is the most frequently proposed instrument in the literature on pecuniary externalities and prudential policy responses, (Bianchi, 2011; Dávila and Korinek, 2018; Jeanne and Korinek, 2019, 2020). This literature typically presumes rational expectations.

In our model, this policy corresponds to the system of linear corrective taxes in the case of homogeneous beliefs (see part (ii) of Proposition 6). This is when investors feature rational expectations, and the social planner optimally taxes borrowing, while any correction of investment decisions is ineffective. Figure 3 displays the welfare effects of this policy in comparison to the non-linear tax system from above.

We employ consumption equivalents relative to the unconstrained competitive allocation, which is when no policy intervention is required, as an ex ante social welfare measure.

In Figure 3, the solid line depicts consumption equivalents of allocations with non-linear corrective taxes, while the dotted line refers to allocations with linear corrective taxes. Each point on the x-axis indicates a specific belief distribution, with beliefs becoming increasingly heterogeneous along the axis.

We find significant welfare gains from non-linear policies over linear Pigouvian taxes. The planner's intervention contains welfare losses at a level of about four to six percent relative to the unconstrained economy. However, if linear taxes are applied to a heterogeneous population, welfare is well below. Corresponding allocations result in welfare losses which are by up to 14 percent larger than compared to allocations with non-linear policies.

Probability of financial distress. The last numerical exercise we provide is related to the above evaluation how probable financial distress is in the competitive equilibrium. We have found that belief disagreements across investors do indeed raise the probability that financial distress occurs, relative to the case of rational and homogeneous beliefs. We repeat the simulation from above, but further account for the constrained-efficient allocation. To that end, we first define the lowest possible realization of R such that collateral constraints in the constrained-efficient allocation are slack.

Definition 6. *Let* $\hat{R}^{**}_{het} \equiv \min\left\{\hat{R} \mid \tilde{\eta}^j_1 = 0 \text{ for all } j\right\}$ *and* $\hat{R}^{**}_{hom} \equiv \min\left\{\hat{R} \mid \tilde{\eta}_1 = 0\right\}$ *be the lowest possible realizations of* R *such that the borrowing constraints are slack in the constrained-efficient equilibrium if investors hold heterogeneous beliefs* $\mathcal{F}$ *or the homogeneous belief* F*, respectively.*

Figure 4 illustrates the mapping from the realization $\hat{R}$ to q_1 for both the competitive and the constrained-efficient equilibrium. The probability of financial distress is indeed lower under constrained efficiency than in the competitive equilibrium. By manipulating investors' behavior through non-linear taxes, the social planner manages to reduce the thresholds of $\hat{R}$, implying that financial distress in the constrained-efficient equilibrium would only arise in markedly unfavorable states. Our previous finding that financial distress is generally less likely under the homogeneous belief than under heterogeneity is further robust to the planner intervention.

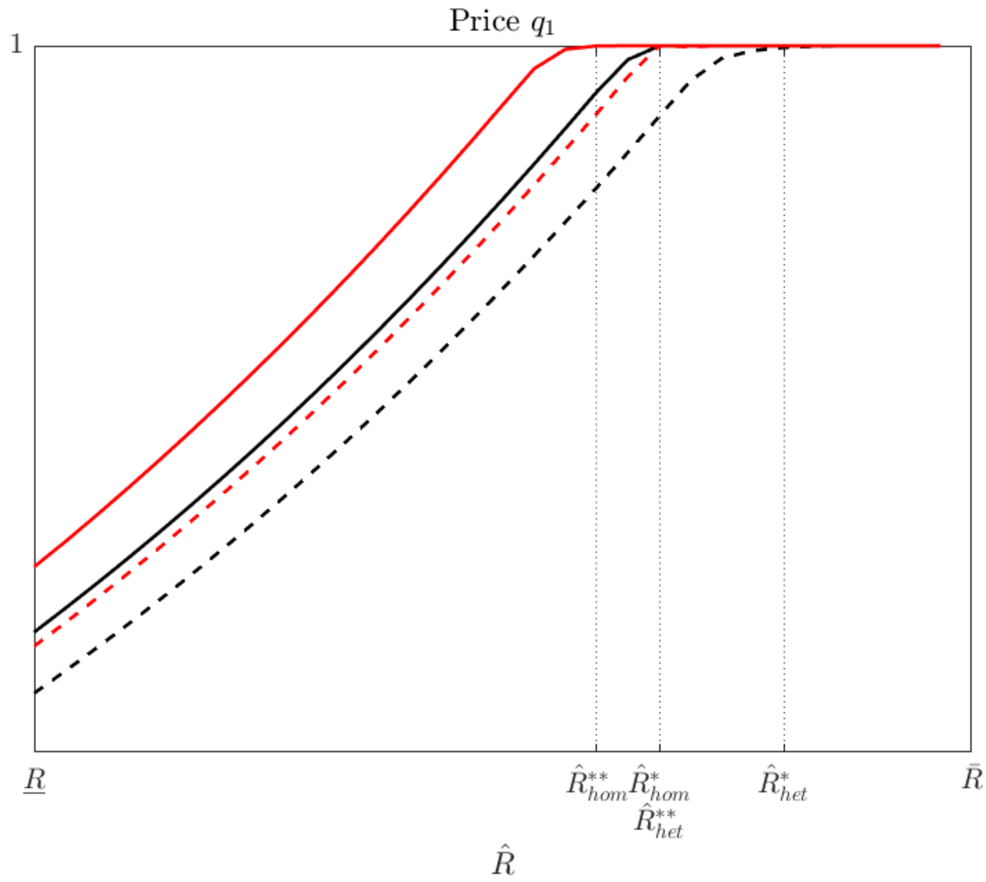

Figure 4: **Mapping from $\hat{R}$ to q_1**

Note: This figure shows the mapping from $\hat{R}$ to q_1 for the two cases when investors hold the homogeneous belief $F(R)$ or heterogeneous beliefs $\mathcal{F}$, respectively. Solid lines refer to the homogeneous case, and dashed lines refer to the heterogeneous case. Black lines refer to the competitive equilibrium, and red lines refer to the constrained-efficient equilibrium. $\hat{R}^{*}_{hom}$, $\hat{R}^{*}_{het}$, $\hat{R}^{**}_{hom}$, and $\hat{R}^{**}_{het}$ are thresholds as defined in Definitions 3 and 6.

5 Final Remarks

This paper presents a theoretical framework to study the contributions of economic agents to financial distress, being the basis on which optimal Pigouvian policies are designed. We build on a model incorporating financial frictions, and enrich it by the heterogeneity of beliefs across economic agents. The framework is employed to analyze the competitive equilibrium, its sensitivity to changes in the underlying set of beliefs, as well as its efficiency properties. We derive optimal corrective policies, which are furthermore quantified in a numerical application.

Our analysis puts forward three key findings. First, we show that, conditional on their beliefs, investors make differentiated contributions to financial distress, where relatively more optimistic agents have an overproportional and decreasing impact on the collateral price. Second, it turns out that financial distress is generally more likely in an economy populated by agents with heterogeneous beliefs, compared to the homogeneous case. Third, we find that a constrained-efficient allocation can be implemented through a system of non-linear Pigouvian taxes, which proves to generate considerable welfare gains over the linear policy advocated by previous articles.

These results add to the literature on financial crises in several ways. We characterize explicitly how financial market participants contribute to distress states. Moreover, in our setting, financial constraints may be binding through ex ante return expectations sufficiently off the ex post realization. This differs from former studies, focusing on financial distress in response to aggregate shocks to investment or net worth. Hence, our framework formalizes a further source of financial distress. Ultimately, our policy proposal improves on linear Pigouvian taxes in an economy featuring heterogeneity of beliefs. The latter point is especially relevant when studying optimal financial regulation in booms and busts, which typically go along with high belief divergence and fluctuations.

Our work lays the ground for further research. Whereas we study optimal ex ante policies in a prudential sense, it may be worthwhile examining optimal ex post policies, such as central bank liquidity injections, under belief heterogeneity. In addition, several types of financial frictions are considered in the literature on prudential policies. The collateral constraints used in this paper link debt issuance to market-valued collateral. However, pecuniary externalities and corrective policies have further been studied in environments with flow constraints, relating to household income or firm cash flows. Their interaction with belief disagreements must still be examined. Ultimately, our three period model may be extended to a dynamic framework, allowing for a more profound quantitative exploration of the effects documented in this paper.

Appendix

Proof of Proposition 1

Models with price-dependent collateral constraints like ours bear the risk that equilibrium prices do not exist. The reason is that these models face downward-sloping supply functions. Constraint agents must sell more if the collateral price is low, but less if it high, and the constraint is less tight.

Existence. We first prove the existence of the equilibrium price. Let

$$S(q_1) = \sum_{j=1}^{J} \mathbb{1}_{\left\{\tilde{l}_1^j(q_1)>0\right\}} s^j \tilde{l}_1^j(q_1) \tag{20}$$

denote the supply of claims as a function of q_1. Analogously, define demand as

$$D(q_1) = -\sum_{j=1}^{J} \mathbb{1}_{\left\{\tilde{l}_1^j(q_1)<0\right\}} s^j \tilde{l}_1^j(q_1). \tag{21}$$

Let $D(q_1)$ and $S(q_1)$ be continuous and differentiable functions on the interval $(0,1]$. Note that $S(q_1)$ is bounded from above for any q_1. This follows from the fact that investors cannot sell more claims than they possess, i.e. $\tilde{l}_1^j \leq \tilde{a}_0^j$, and, hence, for any q_1

$$S(q_1) = \sum_{j=1}^{J} \mathbb{1}_{\left\{\tilde{l}_1^j(q_1)>0\right\}} s^j \tilde{l}_1^j(q_1) \leq \sum_{j=1}^{J} s^j \tilde{a}_0^j = \bar{a}.$$

Specifically, it follows that $\lim_{q_1\to 0} S(q_1) \leq \bar{a}$.

We consider two cases when characterizing the demand curve. First, if demand is zero, there is still excess supply. According to the Walrasian equilibrium definition, all prices q_1 are equilibrium prices.

Second, if demand is positive, we ensure the existence of an equilibrium price q_1 by showing that demand is infinite as the price approaches zero, i.e. $\lim_{q_1\to 0} D(q_1) = \infty$. First, note that buyers will exhaust their entire borrowing limit as they trade, i.e. $\tilde{d}_1^j = \phi q_1 \left(\tilde{a}_0^j - \tilde{l}_1^j\right)$, because any price $q_1 < 1$ grants them a pecuniary benefit. From the period-2 budget constraint (2), we obtain

$$\tilde{c}_2^j = (1-\phi q_1)\left(\tilde{a}_0^j - \tilde{l}_1^j\right). \tag{22}$$

Suppose the price approaches its lower limit of zero, i.e. $q_1 \to 0$. From the price equation (6), it follows that either the numerator tends to zero, i.e. $u'\left(\tilde{c}_2^j\right) \to 0$, or the denominator tends to infinity, i.e. $(1-\phi)u'\left(\tilde{c}_1^j\right)+\phi u'\left(\tilde{c}_2^j\right)\rightarrow\infty$, or both.

If the numerator tends to zero, the concavity of $u\left(\tilde{c}_t^j\right)$ implies that $\tilde{c}_2^j$ becomes infinitely large, i.e. $\tilde{c}_2^j\rightarrow\infty$, and, by (22), so does the demand for claims, i.e. $\tilde{l}_1^j\rightarrow-\infty$.

If, in contrast, the denominator tends to infinity, this can be caused by consumption in $t=1$ and $t=2$ approaching zero, i.e. either $\tilde{c}_1^j\rightarrow 0$ or $\tilde{c}_2^j\rightarrow 0$. In the first case, all consumption is shifted to the final period, i.e. $\tilde{c}_2^j\rightarrow\infty$, from which an infinite demand for claims, i.e. $\tilde{l}_1^j\rightarrow-\infty$, follows again. In the second case, both numerator and denominator of the pricing equation (6) would tend to infinity, yet the numerator at a faster pace as $\phi<1$, and, consequently, the assumption $q_1\rightarrow 0$ would be violated.

Thus, at the minimum price of $q_1\rightarrow 0$, period-2 consumption $\tilde{c}_2^j$ will tend to infinity and $\tilde{l}_1^j$ will tend to minus infinity for all j with $\tilde{l}_1^j<0$. We conclude that overall demand for claims becomes infinitely large, i.e. $\lim_{q_1\rightarrow 0} D\left(q_1\right)=\infty$.

All in all, for $q_1\rightarrow 0$, we obtain a bounded supply and an infinitely high demand. It is only required to ensure that this demand exists. We ensure a positive mass of $D(0)$ through assuming that at least one type of investors has had correct expectations ex-post, receiving a return that is as high as expected or higher. Formally, $E^j[R]\leq\hat{R}$ for at least one j and all realizations $\hat{R}$ of R ensures that there is at least one group that has sufficient funds available in period $t=1$ to demand claims on the asset.

There are different possibilities how supply and demand can intersect. Either $D\left(q_1\right)$ and $S\left(q_1\right)$ intersect on $(0,1]$ at (possibly multiple) price(s). Then, all prices in this set are equilibrium prices. Or they do not have an intersection on the interval. We have shown that, in this case, demand is permanently larger than supply, i.e. $D\left(q_1\right)>S\left(q_1\right)$ for any $q_1\in(0,1]$ as $D(0)>S(0)$ and there is not intersection on $(0,1]$. Hence, $q_1=1$ is the equilibrium price since, for this price, buying investors are indifferent between all levels of feasible demand, and the bounded supply $S(1)<D(1)$ can be fully met. In conclusion, we have shown that the equilibrium price exists.

Uniqueness. Second, we prove that the equilibrium price is unique and satisfies $q_1\leq 1$ in the case of positive demand. Uniqueness is ensured if, first, $\lim_{q_1\rightarrow 0} D\left(q_1\right)=\infty$, second, $D\left(1\right)=S\left(1\right)=0$, and third, if $D\left(q_1\right)$ and $S\left(q_1\right)$ are monotonically decreasing functions on $(0,1]$ with $\frac{\partial D(1)}{\partial q_1}=\frac{\partial S(1)}{\partial q_1}=0$. We continue assuming their continuity and differentiability.

Regarding the first two conditions, we have shown $\lim_{q_1\rightarrow 0} D\left(q_1\right)=\infty$ in the previous part, and $D\left(1\right)=S\left(1\right)=0$ follows from our assumption $\tilde{l}_1^j(1)=0$ for all j.

Next, we prove that both supply and demand are monotone functions on $(0,1]$. Specifically, we determine the signs of

$$\frac{\partial S(q_1)}{\partial q_1} = \sum_{j=1}^{J} \mathbb{1}_{\left\{\tilde{l}_1^j(q_1)>0\right\}} s^j \frac{\partial \tilde{l}_1^j}{\partial q_1} \tag{23}$$

$$\frac{\partial D(q_1)}{\partial q_1} = -\sum_{j=1}^{J} \mathbb{1}_{\left\{\tilde{l}_1^j(q_1)<0\right\}} s^j \frac{\partial \tilde{l}_1^j}{\partial q_1}. \tag{24}$$

Using the period-1 equilibrium conditions (1), (2), (3), and (9), and applying the implicit function theorem to (5), we obtain

$$\frac{\partial \tilde{l}_1^j}{\partial q_1} = \frac{1}{1+(1-2\phi)\,q_1}\left[\frac{1}{(1-\phi q_1)\,A q_1} - 2\phi \tilde{a}_0^j + (2\phi-1)\tilde{l}_1^j\right] \tag{25}$$

Inserting (25) into (23) and (24) yields

$$\frac{\partial S(q_1)}{\partial q_1} = \frac{1}{1+(1-2\phi)\,q_1}\left[\frac{J^S}{(1-\phi q_1)\,A q_1} - 2\phi \sum_{j=1}^{J} \mathbb{1}_{\left\{\tilde{l}_1^j(q_1)>0\right\}} s^j \tilde{a}_0^j + (2\phi-1) S(q_1)\right] \tag{26}$$

$$\frac{\partial D(q_1)}{\partial q_1} = -\frac{1}{1+(1-2\phi)\,q_1}\left[\frac{J^D}{(1-\phi q_1)\,A q_1} - 2\phi \sum_{j=1}^{J} \mathbb{1}_{\left\{\tilde{l}_1^j(q_1)<0\right\}} s^j \tilde{a}_0^j + (2\phi-1) D(q_1)\right], \tag{27}$$

where J^S and J^D are the number of types that are on the supply and the demand side of the market, respectively. We assume that the margin requirement is sufficiently tight, i.e. $\phi < 1/2$.

We first show that the supply curve is a weakly decreasing function of q_1. Recall that $S(q_1)$ is continuous on $(0,1]$, $\lim\limits_{q_1 \to 1} S(q_1) = 0$ and an equilibrium with positive demand $D(q_1) > 0$ requires that there is a q_1 such that $S(q_1) > 0$. Hence, there must further be a $q_1^* \equiv \min\left\{q_1 \mid \frac{\partial S(q_1)}{\partial q_1} < 0 \text{ for all } q_1 > q_1^*\right\}$.

Now we distinguish two cases. If $\frac{\partial S(q_1^*)}{\partial q_1} \neq 0$, there is no $q_1 < q_1^*$ such that $\frac{\partial S(q_1)}{\partial q_1} > 0$, and it follows $\frac{\partial S(q_1)}{\partial q_1} \leq 0$ for all $q_1 \in (0,1]$, making the supply curve monotonically decreasing.

If, however, $\frac{\partial S(q_1^*)}{\partial q_1} = 0$, this is equivalent to $S(q_1^*) = \frac{1}{2\phi-1}\left[2\phi \sum_{j=1}^{J} \mathbb{1}_{\left\{\tilde{l}_1^j(q_1^*)\geq 0\right\}} s^j \tilde{a}_0^j - \frac{J^S}{(1-\phi q_1^*)A q_1^*}\right]$. For $q_1 < q_1^*$, we prove by contradiction that supply is constant.

First suppose that $\frac{\partial S(q_1)}{\partial q_1} > 0$. From (26), it follows that $S(q_1) > S(q_1^*)$ in this case, which would imply $\frac{\partial S(q_1)}{\partial q_1} < 0$, violating the assumption. Now suppose that $\frac{\partial S(q_1)}{\partial q_1} < 0$. From (26), it follows that $S(q_1) < S(q_1^*)$ in this case, which would imply $\frac{\partial S(q_1)}{\partial q_1} > 0$, violating the assumption.

Therefore, we obtain $\frac{\partial S(q_1)}{\partial q_1} = 0$ for all $q_1 < q_1^*$. The constancy of supply for low collateral prices reflects the fact that supply is bounded from above by the amount invested in $t = 0$.

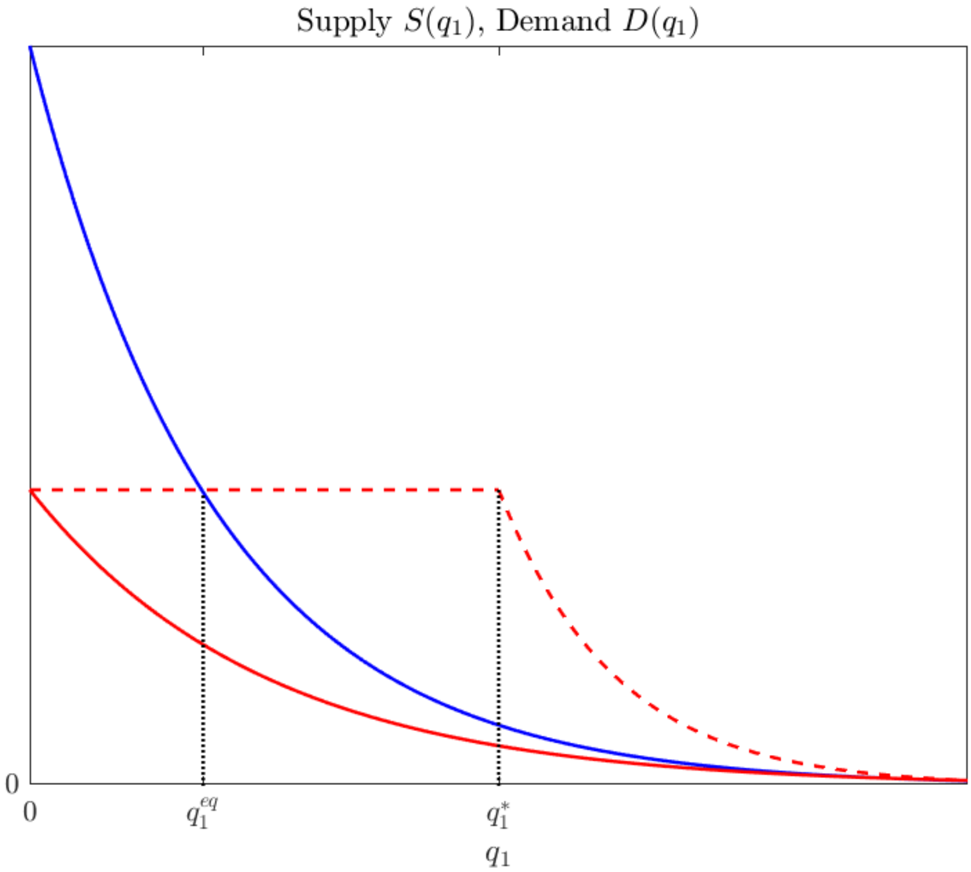

Figure 5: **Supply and demand in $t = 1$**

This figure sketches two possible supply curves and a demand curve in period $t = 1$. Supply curves are depicted in red, while the demand curve is depicted in blue. q_1^{eq} is the equilibrium price, and q_1^* is defined as in the proof of Proposition 1.

q_1^* is thus the price below which distressed investors are willing to liquidate their entire position.

The slope of the demand curve, i.e. the sign of the left-hand side of equation (27), is determined by the term in brackets. Under the assumption of $\phi < 1/2$, and restricting the initial endowment to $\bar{a} \leq 2$, the term in brackets is positive, yielding $\frac{\partial D(q_1)}{\partial q_1} < 0$ for any $q_1 \in (0, 1]$.

Lastly, equations (26) and (27) reveal that $\frac{\partial D(1)}{\partial q_1} = \frac{\partial S(1)}{\partial q_1} = 0$ because $J^S = J^D = \mathbb{1}_{\{\tilde{l}_1^j(1)>0\}} = \mathbb{1}_{\{\tilde{l}_1^j(1)<0\}} = S(1) = D(1) = 0$ at $q_1 = 1$.

Since all the conditions for uniqueness are satisfied, we deduce that the equilibrium price is unique (see Figure 5 for illustration).

Equivalences. Third, we show the two equivalences in part (ii). For part (i), suppose $q_1 = 1$. Combining equations (4) and (5) yields $\tilde{\eta}_1^j = \tilde{\eta}_1^j \phi$. The only solution for the latter condition is $\tilde{\eta}_1^j = 0$. Now, suppose $\tilde{\eta}_1^j = 0$. Equation (4) then becomes $u'\left(\tilde{c}_1^j\right) = u'\left(\tilde{c}_2^j\right)$. Substituting out $u'\left(\tilde{c}_2^j\right)$ in equation (5) yields $q_1 = 1$.

For part (ii), the equivalence is shown formally:

$$\begin{aligned} q_1 &= \frac{u'\left(\tilde{c}_2^j\right)}{(1-\phi)u'\left(\tilde{c}_1^j\right) + \phi u'\left(\tilde{c}_2^j\right)} < 1 \\ \iff &\ (1-\phi)u'\left(\tilde{c}_2^j\right) < (1-\phi)u'\left(\tilde{c}_1^j\right) \\ \iff &\ 0 < u'\left(\tilde{c}_1^j\right) - u'\left(\tilde{c}_2^j\right) = \tilde{\eta}_1^j. \end{aligned}$$

□

Proof of Proposition 2

For the proof of part (i), recall that the period-1 equilibrium price satisfies equation (6), where $\tilde{c}_1^j$ and $\tilde{c}_2^j$ are given by equations (1) and (2) for all j. Since the equilibrium price equals one if $\tilde{\eta}_1^j = 0$, we restrict ourselves to price effects in the case of $\tilde{\eta}_1^j > 0$. For the borrowing constraint to be binding, assume that the realization $\hat{R}$ is sufficiently adverse, satisfying $\hat{R} < 1$. Using CARA $A = -\frac{u''\left(\tilde{c}_t^j\right)}{u'\left(\tilde{c}_t^j\right)}$ for all j and t, we obtain the following equilibrium price derivatives:

$$\frac{\partial q_1}{\partial \tilde{a}_0^j} = \frac{(1-\phi)(1-R)\left(q_1\right)^2}{\frac{u'\left(\tilde{c}_2^j\right)}{u''\left(\tilde{c}_1^j\right)} + (1-\phi)\left(q_1\right)^2 \tilde{l}_1^j} \tag{28}$$

$$\frac{\partial q_1}{\partial \tilde{d}_0^j} = \frac{(1-\phi)\left(q_1\right)^2}{\frac{u'\left(\tilde{c}_2^j\right)}{u''\left(\tilde{c}_1^j\right)} + (1-\phi)\left(q_1\right)^2 \tilde{l}_1^j}. \tag{29}$$

The numerators of equations (28) and (29) are positive, and the denominator is negative. To see this, note that $\frac{\partial q_1}{\partial \tilde{c}_1^j} = -(1-\phi)\frac{u''\left(\tilde{c}_1^j\right)}{u'\left(\tilde{c}_2^j\right)}\left(q_1\right)^2 > 0$. For the denominator, it follows

$$\begin{aligned} &\frac{u'\left(\tilde{c}_2^j\right)}{u''\left(\tilde{c}_1^j\right)} + (1-\phi)\left(q_1\right)^2 \tilde{l}_1^j \leq 0 \\ \iff &\ 1 \geq \frac{\partial q_1}{\partial \tilde{c}_1^j}\tilde{l}_1^j, \end{aligned} \tag{30}$$

which is always satisfied. If $\tilde{l}_1^j \leq 0$, the left-hand side of (30) is negative. But it is exceeded by one even if $\tilde{l}_1^j > 0$. The reason is that $1 \geq \frac{\partial q_1}{\partial \tilde{c}_1^j}\tilde{l}_1^j$ is the condition for finite consumption $\tilde{c}_1^j$. Consider the period-1 budget constraint $\tilde{c}_1^j = R\tilde{a}_0^j + q_1\tilde{l}_1^j + \tilde{d}_1^j - \tilde{d}_0^j$. Increasing the budget by one unit of the consumption good has two effects. First, it directly increases consumption by one unit. Second, it raises q_1, and further increases consumption by $\frac{\partial q_1}{\partial \tilde{c}_1^j}\tilde{l}_1^j$.

Suppose $1 < \frac{\partial q_1}{\partial \tilde{c}_1^j} \tilde{l}_1^j$. In this case, the latter effect via q_1 dominates the direct effect, and the initial stimulus initiated an upward loop towards infinite consumption $\tilde{c}_1^j$. Hence, a finite solution requires $1 \geq \frac{\partial q_1}{\partial \tilde{c}_1^j} \tilde{l}_1^j$, concluding the proof of part (i).

Turning to part (ii), for the equilibrium price derivative with respect to borrowing under a homogeneous belief, we obtain

$$\frac{\partial q_1}{\partial \tilde{d}_0} = \frac{(1-\phi)\,(q_1)^2}{\frac{u'(\tilde{c}_2)}{u''(\tilde{c}_1)}}, \tag{31}$$

which is negative for a concave utility function. □

Proof of Proposition 3

For the proof of part (i), let investors hold heterogeneous beliefs $\mathcal{F}$. The individual type-j decisions for investment and borrowing are governed by equations (8) and (9), that we rewrite as functions of its belief $F^j(R)$ in the following way:

$$q_0 u'\left(\tilde{c}_0^j\left(\tilde{a}_0^j\left(F^j(R)\right), \tilde{d}_0^j\left(F^j(R)\right)\right)\right) = \int_{\underline{R}}^{\overline{R}} R u'\left(\tilde{c}_1^j\left(\tilde{a}_0^j\left(F^j(R)\right), \tilde{d}_0^j\left(F^j(R)\right)\right)\right) \ldots$$
$$\ldots + u'\left(\tilde{c}_2^j\left(\tilde{a}_0^j\left(F^j(R)\right)\right)\right) + \tilde{\eta}_1^j\left(\tilde{a}_0^j\left(F^j(R)\right), \tilde{d}_0^j\left(F^j(R)\right)\right)\phi q_1 dF^j(R) \tag{32}$$
$$u'\left(\tilde{c}_0^j\left(\tilde{a}_0^j\left(F^j(R)\right), \tilde{d}_0^j\left(F^j(R)\right)\right)\right) = \int_{\underline{R}}^{\overline{R}} u'\left(\tilde{c}_1^j\left(\tilde{a}_0^j\left(F^j(R)\right), \tilde{d}_0^j\left(F^j(R)\right)\right)\right) dF^j(R). \tag{33}$$

Notably, period-0 choices $\tilde{a}_0^j\left(F^j(R)\right)$ and $\tilde{d}_0^j\left(F^j(R)\right)$ are direct functions of type j's belief, while period-1 and period-2 variables are both indirect functions of $F^j(R)$ via $\tilde{a}_0^j\left(F^j(R)\right)$ and $\tilde{d}_0^j\left(F^j(R)\right)$ direct functions of it through the expectation operator.

In the following, we apply the calculus of variation, as explained in the main text. Consider a perturbation to beliefs of the form $F^j(R)+\epsilon G^j(R)$, where $\epsilon > 0$ is an arbitrary number, and $G^j(R)$ captures the direction of the perturbation. $F^j(R)+\epsilon G^j(R)$ is required to be a valid cdf for small enough ϵ, so we assume it is continuous and differentiable, it satisfies $G(\underline{R}) = G(\overline{R}) = 0$, and $\partial\left(F^j(R)+\epsilon G^j(R)\right)/\partial R \geq 0$ for sufficiently small ϵ. Lastly, let δ denote the operator for functional derivatives.

We characterize the variational derivatives of investment and borrowing choices when beliefs $F^j(R)$ are perturbed with direction $G^j(R)$, i.e. $\frac{\delta \tilde{a}_0^j}{\delta F^j} \cdot G^j$ and $\frac{\delta \tilde{d}_0^j}{\delta F^j} \cdot G^j$. Optimism and pessimism are measured relative to each other in the sense of first order stochastic dominance. A perturbation $G^j(R)$ makes type-j investors more optimistic if and only if it satisfies $F^j(R)+\epsilon G^j(R) \leq F^j(R)$ for all R. It is easy to see that more optimism requires the perturbation to have a negative direction, i.e. $G^j(R) \leq 0$ for all R. Analogously, investors of type j are made more pessimistic through a perturbation with direction $G^j(R) \geq 0$ for all R.

Applying the implicit function theorem to (32) and (33), and combining the resulting expressions yield

$$
\begin{aligned}
\frac{\delta \tilde{a}_0^j}{\delta F^j} \cdot G^j = & \frac{\int_{\underline{R}}^{\overline{R}} u''\left(\tilde{c}_1^j\right) \tilde{a}_0^j G^j(R) dR \cdot \left(\int_{\underline{R}}^{\overline{R}} (1+\phi) u''\left(\tilde{c}_1^j\right) dF^j(R) + q_0 u''\left(\tilde{c}_0^j\right)\right)}{\left(\int_{\underline{R}}^{\overline{R}} R u''\left(\tilde{c}_1^j\right) dF^j(R) + q_0 u''\left(\tilde{c}_0^j\right)\right) \cdot \left(\int_{\underline{R}}^{\overline{R}} (1+\phi) u''\left(\tilde{c}_1^j\right) dF^j(R) + q_0 u''\left(\tilde{c}_0^j\right)\right)} \cdots \\
& \cdots \frac{-\int_{\underline{R}}^{\overline{R}} \left(u'\left(\tilde{c}_1^j\right) + (R+\phi q_1) u''\left(\tilde{c}_1^j\right) \tilde{a}_0^j\right) G^j(R) dR \cdot \left(\int_{\underline{R}}^{\overline{R}} u''\left(\tilde{c}_1^j\right) dF^j(R) + u''\left(\tilde{c}_0^j\right)\right)}{-\left(\int_{\underline{R}}^{\overline{R}} (R+\phi q_1) R u''\left(\tilde{c}_1^j\right) + (1-\phi q_1) u''\left(\tilde{c}_2^j\right) dF^j(R)\right) \cdot \left(\int_{\underline{R}}^{\overline{R}} u''\left(\tilde{c}_1^j\right) dF^j(R) + u''\left(\tilde{c}_0^j\right)\right)}
\end{aligned}
\tag{34}
$$

$$
\frac{\delta \tilde{d}_0^j}{\delta F^j} \cdot G^j = \frac{-\int_{\underline{R}}^{\overline{R}} u''\left(\tilde{c}_1^j\right) \tilde{a}_0^j G^j(R) dR}{u''\left(\tilde{c}_0^j\right) + \int_{\underline{R}}^{\overline{R}} u''\left(\tilde{c}_1^j\right) dF^j(R)} + \frac{\int_{\underline{R}}^{\overline{R}} R u''\left(\tilde{c}_1^j\right) dF^j(R) + q_0 u''\left(\tilde{c}_0^j\right)}{u''\left(\tilde{c}_0^j\right) + \int_{\underline{R}}^{\overline{R}} u''\left(\tilde{c}_1^j\right) dF^j(R)} \cdot \frac{\delta \tilde{a}_0^j}{\delta F^j} \cdot G^j. \tag{35}
$$

First, we further investigate equation (34). Assuming that the choice of parameters ensures a non-zero trading volume, i.e. $A < 1$ and beliefs $\mathcal{F}$ sufficiently divergent such that $\bar{a} - \tilde{a}_0^j \neq 0$ for some j, and that the borrowing constraints bind in response to the adverse shock, i.e. $\hat{R} < 1$ and $\phi < \frac{1}{2}$ such that $\tilde{\eta}_1^j > 0$ for all j, the numerator is negative for $G^j(R) \leq 0$, and positive for $G^j(R) \geq 0$. The denominator is always negative. Hence, the functional derivative $\frac{\delta \tilde{a}_0^j}{\delta F^j} \cdot G^j$ is positive for $G^j(R) \leq 0$ and negative for $G^j(R) \geq 0$.

Given the signs of the components in (35), it follows that $\frac{\delta \tilde{a}_0^j}{\delta F^j} \cdot G^j$ and $\frac{\delta \tilde{d}_0^j}{\delta F^j} \cdot G^j$ have the same sign for each $G^j(R)$. Consequently, the two variational derivatives in (34) and (35) turn out to be positive if investors are more optimistic ($G^j(R) \leq 0$), and negative if they are more pessimistic ($G^j(R) \geq 0$).

Proving part (ii), we employ the identical procedure as above. Let investors hold the homogeneous belief $F(R)$. Let further $G(R)$ be the direction of a perturbation of the homogeneous belief. We obtain as the functional derivative of borrowing

$$
\frac{\delta \tilde{d}_0}{\delta F} \cdot G = \frac{-\int_{\underline{R}}^{\overline{R}} u''\left(\tilde{c}_1\right) \bar{a} G(R) dR}{u''\left(\tilde{c}_0\right) + \int_{\underline{R}}^{\overline{R}} u''\left(\tilde{c}_1\right) dF(R)}, \tag{36}
$$

which is as well positive for more optimistic investors ($G^j(R) \leq 0$), and negative for more pessimistic investors ($G^j(R) \geq 0$). □

Proof of Theorem 1

With regard to part (i), let investors hold heterogeneous beliefs $\mathcal{F}$. Let further $G^j(R)$ be the direction of a perturbation of type-j investors' belief $F^j(R)$, and beliefs $F^i(R)$ be constant for all $i \neq j$.

Recall that the functional derivative $\frac{\delta}{\delta F^j} \cdot G^j$ describes a gradient, so it is identical to a partial derivative if the functional argument is one-dimensional. We write the period-1 equilibrium price as a function of beliefs, i.e. $q_1 = q_1\Big(\tilde{a}_0(\mathcal{F}), \tilde{d}_0(\mathcal{F})\Big)$. It follows

$$\frac{\delta q_1}{\delta F^j} \cdot G^j = \frac{\delta q_1}{\delta \tilde{a}_0^j} \cdot \frac{\delta \tilde{a}_0^j}{\delta F^j} \cdot G^j + \frac{\delta q_1}{\delta \tilde{d}_0^j} \cdot \frac{\delta \tilde{d}_0^j}{\delta F^j} \cdot G^j = \frac{\partial q_1}{\partial \tilde{a}_0^j} \cdot \frac{\delta \tilde{a}_0^j}{\delta F^j} \cdot G^j + \frac{\partial q_1}{\partial \tilde{d}_0^j} \cdot \frac{\delta \tilde{d}_0^j}{\delta F^j} \cdot G^j. \tag{37}$$

Using Propositions 2 and 3, we obtain statement (1) of part (i).

For statement (2), let $G^j(R) < 0 < G^i(R)$ with $|G^j(R)| = |G^i(R)|$ for all R be the directions of two perturbations that make investors of type j more optimistic, and investors of type i more pessimistic by the same magnitude. We investigate each factor in the two summands on the right-hand side of equation (37) separately. First, note that equations (34) and (35) imply that

$$\left|\frac{\delta \tilde{a}_0^j}{\delta F^j} \cdot G^j\right| = \left|\frac{\delta \tilde{a}_0^i}{\delta F^i} \cdot G^i\right| \text{ and } \left|\frac{\delta \tilde{d}_0^j}{\delta F^j} \cdot G^j\right| = \left|\frac{\delta \tilde{d}_0^i}{\delta F^i} \cdot G^i\right|.$$

Second, taking the derivatives of equations (28) and (29) shows that q_1 is a (decreasing and) concave function of investment and borrowing, i.e. $\frac{\partial^2 q_1}{\partial^2 \tilde{a}_0^j} \leq 0$ and $\frac{\partial^2 q_1}{\partial^2 \tilde{d}_0^j} \leq 0$. As for any concave function, it follows that

$$\left|\frac{\delta q_1}{\delta \tilde{a}_0^j}\right| > \left|\frac{\delta q_1}{\delta \tilde{a}_0^i}\right| \text{ and } \left|\frac{\delta q_1}{\delta \tilde{d}_0^j}\right| > \left|\frac{\delta q_1}{\delta \tilde{d}_0^i}\right|.$$

Inserting the two former results in equation (37) yields statement (2).

To prove part (ii), let investors hold the homogeneous belief $F(R)$. Let further $G(R)$ be the direction of a perturbation of the homogeneous belief. Equation (37) simplifies to

$$\frac{\delta q_1}{\delta F} \cdot G = \frac{\partial q_1}{\partial \tilde{d}_0} \cdot \frac{\delta \tilde{d}_0}{\delta F} \cdot G, \tag{38}$$

which is negative for $G(R) \leq 0$ and positive for $G(R) \geq 0$ by the same arguments as in statement (1) of part (i). □

Proof of Theorem 2

We start out by proving that $\hat{R}^*_{het} > \hat{R}^*_{hom}$, where $\hat{R}^*_{het}$ and $\hat{R}^*_{hom}$ are defined in Definition 3.

Consider a population with investors holding heterogeneous beliefs $\mathcal{F}$. Let $\hat{R}^{*j}_{het}$ denote the lowest possible realization $\hat{R}$ such that the collateral constraint of type-j investors is slack, i.e. $\tilde{\eta}_1^j = 0$ and $q_1 = 1$, which are equivalent to $\tilde{c}_1^j = \tilde{c}_2^j$. At this point, the borrowing constraint yields $\tilde{d}_1^j = \phi \tilde{a}_0^j$. Using this, and equating the budget constraints (1) and (2), one obtains $\hat{R}^{*j}_{het} = 1 - 2\phi + \frac{\tilde{d}_0^j}{\tilde{a}_0^j}$.

Given the result from Proposition 1, it suffices that one type of investors is constrained to make all investors constrained. We refer to this situation as financial distress, and it follows that $\hat{R}^*_{het} = \max\left\{\hat{R}^{*j}_{het}\right\}_{j\in\{1,...,J\}}$. Assuming without loss of generality that investors are ordered from more to less optimistic types, i.e. $F^1(R) < ... < F^J(R)$ for all R, we obtain $\hat{R}^*_{het} = \hat{R}^{*1}_{het}$. For the homogeneous case, we derive $\hat{R}^*_{hom} = 1 - 2\phi + \frac{\tilde{d}_0}{\bar{a}}$ equivalently.

To show that $\hat{R}^*_{het} > \hat{R}^*_{hom}$, it is sufficient to prove that $\frac{\tilde{d}^1_0}{\tilde{a}^1_0} > \frac{\tilde{d}_0}{\bar{a}}$. Since type $j = 1$ is the most optimistic type, we know that $\tilde{a}^1_0 > \bar{a}$ and $\tilde{d}^1_0 > \tilde{d}_0$. To prove that $\frac{\tilde{d}^1_0}{\tilde{a}^1_0} > \frac{\tilde{d}_0}{\bar{a}}$, we show that $\tilde{d}^1_0 - \tilde{d}_0 > \tilde{a}^1_0 - \bar{a}$.

The latter statement would follow if a perturbation, making a specific belief more optimistic, i.e. $G^1(R) < 0$ for all R, always increased borrowing by more than investment, i.e. $\frac{\delta\tilde{d}^1_0}{\delta F^1} \cdot G^1 > \frac{\delta\tilde{a}^1_0}{\delta F^1} \cdot G^1$. We deduce from equation (35) that this condition is satisfied provided that

$$\frac{\int_{\underline{R}}^{\overline{R}} Ru''\left(\tilde{c}^1_1\right) dF^1 + q_0 u''\left(\tilde{c}^1_0\right)}{u''\left(\tilde{c}^1_0\right) + \int_{\underline{R}}^{\overline{R}} u''\left(\tilde{c}^j_1\right) dF^1} > 1. \tag{39}$$

Under the presumption made in Theorem 2, requiring the homogeneous belief $F(R)$ to be less optimistic than at least one type's belief in the heterogeneous case, implying $F^1(R) < F(R)$ for all R, inequality (39) is satisfied for any type-1 belief F^1 sufficiently optimistic. Hence, under this assumption, we obtain $\hat{R}^*_{het} > \hat{R}^*_{hom}$.

Ultimately, we derive the corresponding probabilities of financial distress. In our setting, it is for the heterogeneous and the homogeneous case, respectively

$$\Pr\left(\tilde{\eta}^1_1 > 0\right) = \Pr\left(R \leq \hat{R}^*_{het}\right) = F\left(\hat{R}^*_{het}\right)$$
$$\Pr\left(\eta_1 > 0\right) = \Pr\left(R \leq \hat{R}^*_{hom}\right) = F\left(\hat{R}^*_{hom}\right).$$

Given $\hat{R}^*_{het} > \hat{R}^*_{hom}$ and the strict monotonicity of the cdf F, it follows that $F\left(\hat{R}^*_{het}\right) > F\left(\hat{R}^*_{hom}\right)$. □

Proof of Proposition 4

Proposition 4 follows from Definition 4 and Proposition 2. □

Proof of Proposition 5

With regard to part (i), let investors hold heterogeneous beliefs $\mathcal{F}$. Let further $G^j(R)$ be the direction of a perturbation of type-j investors' belief $F^j(R)$, and beliefs $F^i(R)$ be constant for all $i \neq j$. We calculate the functional derivatives of distributive and collateral externalities with respect to beliefs in the following way:

$$\frac{\delta D^i_{\tilde{a}^j_0}}{\delta F^j} \cdot G^j = \frac{\delta\left(\frac{q_1}{\partial \tilde{a}^j_0}\right)}{\delta F^j} \cdot G^j \cdot \tilde{l}^j_1 = \left(\frac{\partial^2 q_1}{\partial \tilde{a}^j_0 \partial \tilde{a}^j_0} \frac{\delta \tilde{a}^j_0}{\delta F^j} \cdot G^j + \frac{\partial^2 q_1}{\partial \tilde{a}^j_0 \partial \tilde{d}^j_0} \frac{\delta \tilde{d}^j_0}{\delta F^j} \cdot G^j\right) \tilde{l}^j_1,$$

and analogously for $D^i_{\tilde{d}^j_0}$, $C^i_{\tilde{a}^j_0}$, and $C^i_{\tilde{d}^j_0}$. Since q_1 is strictly decreasing and concave in both $\tilde{a}^j_0$ and $\tilde{d}^j_0$, and using our results from above on the sign of the functional derivatives $\frac{\delta \tilde{a}^j_0}{\delta F^j} \cdot G^j$ and $\frac{\delta \tilde{d}^j_0}{\delta F^j} \cdot G^j$, it follows that the term in brackets is unambiguously negative for $G^j(R) < 0$, and positive for $G^j(R) > 0$. This proves the first statement of part (i).

Statement (2) of part (i), as well as part (ii), follow from the same arguments as those used in the proof of Theorem 1. □

Proof of Proposition 6

First, we derive the tax formulas in part (i). Consider the period-0 optimization problem of a type-j agent with taxes:

$$\max_{c^j_0, a^j_0 \geq 0, d^j_0} u\left(c^j_0\right) + E^j\left[V^j_1\left(a^j_0, d^j_0 | \tilde{a}_0, \tilde{d}_0\right)\right] \quad \text{s.t.}$$

$$\left(\lambda^j_0\right) \quad c^j_0 = e + \left(1 - \tau^j_d \operatorname{sgn}\left(\tilde{d}^j_0\right)\right) d^j_0 + \left(1 - \tau^j_a \operatorname{sgn}\left(\bar{a} - \tilde{a}^j_0\right)\right) q_0 \left(\bar{a} - a^j_0\right) + T^j. \tag{40}$$

This problem gives rise to the following optimality conditions:

$$\left(1 - \tau^j_a \operatorname{sgn}\left(\bar{a} - \tilde{a}^j_0\right)\right) q_0 u'\left(c^j_0\right) = E^j\left[R u'\left(c^j_1\right) + u'\left(c^j_2\right) + \eta^j_1 \phi q_1\right] \tag{41}$$

$$\left(1 - \tau^j_d \operatorname{sgn}\left(\tilde{d}^j_0\right)\right) u'\left(c^j_0\right) = E^j\left[u'\left(c^j_1\right)\right]. \tag{42}$$

In a symmetric equilibrium, it will always be the case that $c^j_0 = \tilde{c}^j_0$, $a^j_0 = \tilde{a}^j_0$ and $d^j_0 = \tilde{d}^j_0$ for each j. Combining the latter two conditions with their counterparts from the social planner problem, i.e. equations (13) and (14), respectively, using the planner's pricing relation $\tilde{\psi} = q_0 \omega^j u'\left(\tilde{c}^j_0\right)$, and solving for the taxes yields the tax formulas (15) and (16).

Second, it follows that, using these taxes, the competitive allocation is constrained-efficient. Specifically, substituting (15) and (16) into the optimality conditions of the competitive allocation with taxes, i.e. (41), and (42), replicates the planner's optimality conditions (13) and (14), as well as $\tilde{\lambda}_0 = \omega^j u'\left(\tilde{c}^j_0\right)$ for each j. Moreover, rebating revenues through T^j for all j ensures that individual period-0 budget constraints are satisfied, and the same holds for the resource constraint in consequence. To summarize, the competitive allocation with taxes satisfies the identical set of conditions, so it turns out to be constrained-efficient.

By the same arguments, we derive the homogeneous tax formula 17 in part (ii). □

Proof of Theorem 3

Theorem 3 follows from Propositions 3 and 6. □

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