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Strategic Communication with a Small Conflict of Interest*

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Abstract

This paper analyzes strategic information transmission between a sender and a receiver with similar objectives. We provide a first-order approximation of the equilibrium behavior in the general version of the Crawford and Sobel's (1982) model with a small bias. Our analysis goes beyond the usual uniform-quadratic setting: we uncover how the state-dependent bias and the non-uniform state distribution influence the precision with which each state of the world is communicated. We illustrate the approach by providing novel comparative statics results in different applications.

Keywords: Strategic Communication, Small Bias

JEL Classifications: C72, D82, D83

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1 Introduction

The study of strategic information transmission has proven to be useful in analyzing a number of economic phenomena, such as central bank announcements and communication within organizations. The analysis typically focuses, among other aspects, on understanding how different primitives—such as the degree of incentive misalignment or the type of information transmitted—affect the equilibrium communication and the corresponding welfare implications.

Most of the cheap-talk literature (reviewed below) analyzes variations of the uniform-quadratic case of the Crawford and Sobel (1982, henceforth CS) model.[1] The main reason for restricting the focus to the uniform-quadratic case is technical rather than conceptual; the discreteness of the equilibrium construction makes the analysis of more general settings difficult and, in many cases, impossible. Still, in most cases, not only does the restriction engender a significant loss of generality, but it also limits the potential insights that studying strategic communication can provide.

This paper takes a different approach with the aim of deepening our understanding of strategic communication and its implications for equilibrium behavior and welfare. We develop new tools to provide a complete first-order characterization of equilibrium behavior in a general version of the CS model, under the assumption that the conflict of interest is small. We show that equilibrium communication can be approximated by a simple linear differential equation describing how the primitives of the model affect the imprecision with which each state is communicated. We illustrate the usefulness of our approach by providing new comparative statics results in different applications of the CS model.

Our analysis is useful in applications where communication between agents is plausibly not very coarse. For example, institutions and contracts are designed, in part, to improve the objective alignment of different agents, and hence improve the precision of the information transmitted through either internal communication (e.g., intra-firm information transmission) or external communication (e.g., central bank announcements). Reputation mechanisms (not modeled in our setting) can also align the incentives of different agents, and improve communication as a result.

In the general version of the CS model, a sender observes a one-dimensional state of the world $\theta \in [0,1]$ distributed according to some distribution F, sends a cheap-talk message to a receiver, and the receiver takes a one-dimensional action. Their payoff depends only on

[1] Some papers partially relax the uniform-quadratic assumption; see the literature review below.

the state of the world and the action taken. The bias at each state θ—that is, the difference between the ideal actions of the sender and the receiver when state θ is realized—is $\beta\, b(t)$, where $\beta > 0$ is the "size of the bias" and $b(\theta)$ is the "shape of the bias". Like the majority of the previous literature, we focus most (but not all) of our analysis on Pareto efficient equilibria. Our goal is to characterize communication in settings where β is small.

Our first result provides a first-order approximation of the equilibrium information transmission. We characterize the imprecision by which each state of the world θ is communicated relative to the size of the bias.[2] It is measured as the squared length of the equilibrium interval containing it divided by β. When β is small, the imprecision approximates a function $c(\theta)$, called the *coarseness of communication*. Notably, $c(\cdot)$ can be obtained solving a simple first-order linear differential equation. Such an equation establishes that the marginal change of the coarseness of communication at θ is proportional to the difference between two terms. The first term is the shape of the bias at θ, $b(\theta)$, so a positive bias tends to make the information transmission coarser in higher states. Intuitively, the sender does not have the incentive to over-report (when the bias is positive) only if higher states are communicated more coarsely. The second term depends on the distribution, and can be interpreted as an endogenous bias of the receiver. It is given by the deviation of the action he chooses from the center of the equilibrium interval containing θ. This term is positive when high states are more likely; in this case the receiver tends to take higher actions. Therefore, the *net bias*—the difference between the bias of the sender and the endogenous bias of the receiver—determines the direction towards which the equilibrium information transmission becomes less precise. If, for example, the bias is positive and the density function is increasing, different states are communicated similarly precisely.

We next introduce the concept of *communicable state*: a state with zero coarseness of communication. We prove the existence and generic uniqueness of a communicable state, which is not necessarily the state with smallest bias. We express coarseness of communication as a cumulative process that accumulates (and sometimes de-accumulates) as the state moves away from the communicable state. The rate at which communication deteriorates depends on whether the net bias at the state points toward, or in the opposite direction of, where the communicable state. In the first case, communication becomes more coarse at states away from it, while the reverse is true in the second case. The loca-

[2] As first Spector (2000) showed, the CS model admits equilibria with precise information transmission when the preferences of the sender and the receiver are close. Our result pins down both the rate of convergence and the state-dependent precision.

tion of the communicable state then has a crucial role in determining the effect of changes on the bias or the distribution of states.

Comparative statics results. We illustrate our methodology by providing two novel comparative statics results, motivated through some applications of the CS model.

The first result characterizes the effect that reductions in the bias have on how information is transmitted in equilibrium. We ask whether the conventional wisdom that a higher incentive alignment leads to better communication—which is true in the uniform-quadratic case—applies in general. We show that communication is always better, for example, when objectives become uniformly more aligned across states. Sometimes, nevertheless, decreasing the incentive misalignment in some states worsens communication, and we characterize the conditions for this to occur. This happens, for example, when the sender prefers actions closer to an agreement state than the receiver's desired actions, as when a central banker communicates with the private sector. We argue that appointing a new central banker who is equally hawkish on high inflation but more lenient on low inflation may worsen communication and, as a result, reduce the payoff of all agents.

Our second result characterizes the optimal role assignment of biased agents in a communication setting. This corresponds, for example, to the assignment of differently efficient workers to different tasks in a new project, one focussed on obtaining information and the other focussed on deciding the joint effort. The role assignment influences the equilibrium information transmission between them—reversing their roles reverses the sign of the bias of the sender and also the position of the communicable state. We show that, when the bias is monotone, the role assignment that makes bias increasing maximizes the communication efficiency. This has the implication that assigning the worker with a lower cost of effort to be the sender is optimal.

1.1 Literature review

Our paper contributes to the cheap-talk literature on the strategic information transmission between a sender and a receiver, first formalized by Crawford and Sobel (1982). The CS model, which has become the workhorse model of the cheap-talk literature, considers a natural communication setting and characterizes the form of its equilibria. Still, due to its lack of tractability, many papers have focused on their attention in the uniform-quadratic-constant case. Some exceptions are the following. Melumad and Shibano (1991), who show that when the bias is linear, if there is enough disagreement between the sender and the receiver, communication may make the sender worse off. Alonso and Matouschek (2008) shows that, also in the case where the bias is linear, equilibria with an infinite number of messages may exist. Gordon (2010) generalizes the finding to "outward biases" and characterizes the conditions for the existence of equilibria with an infinite number of states. Alternatively, Szalay (2012) studies how changes (spreads/shifts) in the distribution of states affect communication under some restrictions on the payoff and distribution functions. Another strand of the cheap-talk literature allows for a state-dependent bias or a non-uniform distribution of states, while changing the economic problem to be studied. Examples are Admati and Pfleiderer (2004) and Kawamura (2015) (where the receiver is unsure of the sender's competence); Ottaviani (2000) (where the receiver is naive with some probability); Morgan and Stocken (2003), Li and Madarász (2008), Dimitrakas and Sarafidis (2005), and Deimen and Szalay (2019) (where the bias is constant but uncertain); and Deimen and Szalay (2014) (where the sender and the receiver do not know the state of the world and disagree on the relative importance of multiple informative signals). We contribute to this literature by providing new techniques and results to study the basic problem of strategic information transition between two rational agents when their conflict of interest is small, while keeping the model in its general form. The tractability of our approach allows us to provide new comparative statics, which shed light on how the different components of the model shape equilibrium communication and welfare.[3]

The paper is organized as follows. After this introduction, Section 2 presents the base model and some preliminary results on the case where the bias is small. Section 3 provides the analysis and main results of the CS model with a small bias, and Section 4 presents some applications of the model and comparative statics results. Finally, Section 5 discusses the assumption of small bias and concludes. Appendix A contains the proofs of all results in the paper. Appendix B generalizes the results to arbitrary payoff functions and provides a characterization of inefficient equilibria.

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[3]There are other papers where which study cheap-talk models with equilibria where information transmission is very precise. For example, Battaglini (2004), Eso and Fong (2008) and Ambrus and Lu (2014), discuss models where multiple senders imperfectly observe the state and obtain that fully revealing equilibria may exist. Ottaviani and Squintani (2006), Kartik, Ottaviani, and Squintani (2007), Kartik (2009) and Chen (2011), assume that messages are payoff-relevant, effectively transforming the communication problem into a signaling one. Models with noisy communication such as Blume, Board, and Kawamura (2007) contain equilibria with an uncountable number of partitions. Finally, Dilmé (2018) analyzes optimal languages where there is no bias between the sender and the receiver and studies the case where the number of available messages for communication is large.

2 Base model

2.1 Setting

We consider a setting analogous to Crawford and Sobel (1982). There is a sender and a receiver. First, nature draws a state of the world (or type) θ from a set $\Theta \equiv [0,1]$ using an absolutely continuous distribution F with a continuous, strictly positive and differentiable probability density function f with a bounded derivative. The sender observes the state and chooses a message $m \in M$, where M is an infinite set. The receiver only observes the message sent by the sender and then she chooses an action $a \in \mathbb{R}$.

We first focus our analysis on the case where, for each given outcome of the game $(\theta, m, a) \in \Theta \times M \times \mathbb{R}$, the payoff of the sender is $u^s(\theta, a) \equiv -(a - \theta - \beta\ b(\theta))^2$, while the payoff of the receiver is $u^r(\theta, a) \equiv -(a-\theta)^2$, where $\beta > 0$ is the "size of the bias" and $b : \Theta \to \mathbb{R}$ is the "shape of the bias," which is assumed to be Lipschitz-continuous and has a finite or countable number of zeros. Paralleling the assumptions in Crawford and Sobel (1982), we assume that $\theta + \beta\ b(\theta)$ is increasing in θ. This specification allows us to focus our initial attention on how the bias and the state distribution affect equilibrium communication. Appendix B shows that the analysis holds for more general payoff functions.

Definition 2.1. An *(Bayes-Nash) equilibrium* of our game is a pair of functions $\mu : \Theta \to \Delta(M)$ and $\alpha : M \to \Delta(\mathbb{R})$ such that

1. **Sender's Optimality:** $\mu \in \arg\max_{\mu'} \mathbb{E}_{\theta,a}[u^s(\theta,a)|\mu', \alpha]$ and
2. **Receiver's Optimality:** $\alpha \in \arg\max_{\alpha'} \mathbb{E}_{\theta,a}[u^r(\theta,a)|\mu, \alpha']$,

where $\mathbb{E}_{\theta,a}$ refers to the expectation with respect to the variables θ and a.

An equilibrium $\langle \mu, \alpha \rangle$ is a *partition equilibrium* if the positive-measure elements of $\{\mu^{-1}(m) | m \in M\}$ form an interval partition of Θ (each of its elements is called a *partition element*). As Crawford and Sobel (1982) shows, all equilibria are "essentially equivalent" (i.e., generate the same joint distribution of states and actions) to partition equilibria, and we call them just equilibria.[4] We use $[\theta_n, \theta_{n+1})$ to denote a generic partition element of an equilibrium. It will be convenient to use $[\underline{\tau}(\theta), \overline{\tau}(\theta))$ to denote the partition element containing state θ. Finally, $\alpha(\theta_n, \theta_{n+1})$ will denote the equilibrium action of the receiver in $[\theta_n, \theta_{n+1})$, and $\alpha(\theta) \equiv \alpha(\underline{\tau}(\theta), \overline{\tau}(\theta))$ the equilibrium action when the realized state is θ.

[4]For the payoff functions considered in this section, the crucial requirements for this result to hold are that (i) $b(\theta)$ is only zero in a finite set, and (ii) $\theta + \beta\ b(\theta)$ is increasing. See Section B.1 for the conditions in the general case.

3 Communication under small conflict of interest

This section characterizes the outcomes of Pareto-efficient equilibria when the bias is small. Our results are enough to characterize the main properties of equilibrium communication in settings with low objective misalignment between the sender and the receiver.

3.1 Equilibrium construction

Before analyzing the case where the bias is small, we briefly review how equilibria are constructed in the CS model. A standard result is that, for a given strictly increasing tuple $(\theta_n)_{n=0}^N$, the collection $\{[\theta_{n-1},\theta_n)|n=1,...,N\}$ is the set of partition elements for some finite partition equilibrium if and only if $\theta_0=0$, $\theta_N=1$, and for all n there is $\alpha(\theta_n,\theta_{n+1})$ satisfying[5]

$$\alpha(\theta_n,\theta_{n+1}) \in \arg\max_a \int_{\theta_n}^{\theta_{n+1}} \frac{f(\theta)}{F(\theta_{n+1})-F(\theta_n)}\, u^r(\theta,a)\, \mathrm{d}\theta\ , \tag{3.1}$$

$$u^s\big(\theta_n,\alpha(\theta_{n-1},\theta_n)\big) = u^s\big(\theta_n,\alpha(\theta_n,\theta_{n+1})\big)\ . \tag{3.2}$$

The first equation corresponds to the optimality condition of the receiver: she chooses an optimal action given her belief about the realization of the state of the world. The second equation corresponds to the sender's optimality condition: it requires him to be indifferent between the equilibrium actions of two consecutive intervals of the equilibrium partition when the state of the world is their common boundary.

Equations (3.1)-(3.2) aggregate state-dependent properties (likelihood, bias,...) of the states of each of the partition intervals. Furthermore, the boundary requirements ($\theta_0 = 0$ and $\theta_N = 1$) impose a global condition. This makes it difficult, in general, to characterize the solutions of the system of difference equations. Additionally, the discrete nature of the difference equation implies small local changes in the primitives of the model may generate a discontinuous change in the equilibrium outcomes. Our analysis overcomes these difficulties by transforming the system of difference equations into a single differential equation. We obtain explicit solutions of the differential equation, which correspond to approximate solutions of the system of difference equations when the bias is small.

[5]It is convenient for simplicity (and notationally) to focus the arguments in the main text on finite equilibria. Our results also are valid when equilibria with an infinite number of partition elements are considered (their proofs accommodate this case).

We will focus our analysis on studying *efficient equilibria*, that is, equilibria satisfying that there is no other equilibrium where both the sender and the receiver obtain a higher payoff, one of them strictly (a standard argument involving sequences of equilibria shows that such equilibria exist). The reasons for focussing on these equilibria are the following. First, they give us a lower bound on the size of the distortion that strategic communication generates, which have made them the object of study of much of the previous literature.[6] Second, they provide us with a natural way to compare communication for different primitives of the model, given the usual equilibrium multiplicity that communication models have. As we will see, even though there may be multiple efficient equilibria, they all have similar properties when the bias is small. Appendix B generalizes some results to non-efficient equilibria.

3.2 Main result

We begin presenting our main result, which gives an approximation of how coarse communication is in an efficient equilibrium as a function of the primitives of the model:

Proposition 3.1. *There is some $\theta^* \in [0,1]$ such that, for any efficient equilibrium,*

$$\beta^{-1}\,(\overline{\tau}(\theta)-\underline{\tau}(\theta))^2 \approx c(\theta) \equiv \frac{\int_{\theta^*}^{\theta} 8\, b(\theta')\, f(\theta')^{2/3}\, \mathrm{d}\theta'}{f(\theta)^{2/3}} \quad \textit{for all } \theta \in \Theta\,, \tag{3.3}$$

where "≈" means "equal except for terms that vanish as $\beta \to 0$."[7]

Proposition 3.1 achieves a notable feat: it characterizes, in a first-order approximation, how coarsely information is transmitted in an efficient equilibrium with a simple integral expression. The left hand side of equation (3.3) is the square of the length of the interval of states it communicates, that is, it measures how imprecise the information that the message communicating state θ transmits is. On its right hand side we see that, if the bias is small, we can approximate such imprecision with the function $c(\cdot)$, which we call

[6] In the same way that we argue that institutions (such as firms) which benefit from better communication of their agents may try to reduce their bias (so the bias of our model is the residual bias which cannot be eliminated through institutional design), they may also try to induce agents to play equilibria which are not Pareto dominated.

[7] In the statement, "for any efficient equilibrium" means that "for any sequence $(\beta^i)_i$ strictly decreasing to 0, corresponding sequence of efficient equilibria (indexed by i) and $\theta \in \Theta$, $\lim_{i\to\infty}\left(\frac{(\overline{\tau}^i(\theta)-\underline{\tau}^i(\theta))^2}{\beta^i} - c(\theta)\right) = 0$". Also, we use the conventional notation that $\int_x^y = -\int_y^x$ whenever $x > y$.

coarseness of communication (we describe below how θ^* is pinned down).[8] We now explain why Proposition 3.1 holds, and afterwards we exemplify how it can be used to obtain novel economic insights.

Uniform distribution: To obtain an intuition for Proposition 3.1, we first consider the case of a uniform distribution. Fix an equilibrium and consider two consecutive messages with partition elements $[\theta_{n-1},\theta_n)$ and $[\theta_n,\theta_{n+1})$, respectively. If the realized state is θ_n, the sender is indifferent between the actions $a(\theta_{n-1},\theta_n)$ and $a(\theta_n,\theta_{n+1})$, so they must be at the same distance of his ideal action $\theta_n + \beta\, b(\theta_n)$ (recall equation (3.2)). Since the receiver takes the middle action in each of the intervals, it must then be that

$$\overline{\tau}(\theta)+\beta\, b(\theta)-\frac{\underline{\tau}(\theta)+\overline{\tau}(\theta)}{2}=\frac{\overline{\tau}(\theta)+\overline{\overline{\tau}}(\theta)}{2}-\big(\overline{\tau}(\theta)+\beta\, b(\theta)\big) \tag{3.4}$$

for all $\theta\in\Theta$, where $\overline{\overline{\tau}}(\theta)$ indicates the upper bound of the partition element next to $[\underline{\tau}(\theta),\overline{\tau}(\theta))$, and $a^s(\theta)\equiv\theta+\beta\; b(\theta)$ indicates the optimal action for the sender in state θ. The previous expression can be finally rewritten as

$$\beta^{-1}\;\frac{(\overline{\overline{\tau}}(\theta)-\overline{\tau}(\theta))^2-(\overline{\tau}(\theta)-\underline{\tau}(\theta))^2}{\overline{\tau}(\theta)-\underline{\tau}(\theta)}\approx 8\; b(\theta)\quad\text{for all } \theta\in\Theta\,. \tag{3.5}$$

Equation (3.3) (with $f\equiv 1$) follows from the previous expression. Hence, since $c(\theta)\approx\beta^{-1}\,(\overline{\tau}(\theta)-\underline{\tau}(\theta))^2$, we have that (3.5) becomes

$$c'(\theta)=8\; b(\theta)\quad\text{for all } \theta\in\Theta\,. \tag{3.6}$$

This equation is equivalent to equation (3.3).

Non-uniform distribution: We now consider the case where the distribution is not uniform. Fix a small β, an efficient equilibrium, and a state θ. The receiver's optimal action within an equilibrium interval is close to the center of such an interval. Indeed, since f is differentiable, its variation in $[\underline{\tau}(\theta),\overline{\tau}(\theta))$ is small. As a result, since f is also bounded away from 0, the receiver's problem of choosing her optimal action within such a small interval approximates the uniform-quadratic-constant case (where the optimal action is in the middle of the interval). The following proposition shows that the deviation of the receiver's optimal action from the middle point of the interval can be approximated without the need of explicitly computing an efficient equilibrium:

[8] The fact that communication can be precise when the preferences of the sender and the receiver are close was first established in Spector (2000).

Proposition 3.2. *For any efficient equilibrium,*

$$\beta^{-1}\left(\alpha(\theta)-\frac{\overline{\tau}(\theta)+\underline{\tau}(\theta)}{2}\right)\approx\underbrace{\frac{1}{12}\,\frac{f'(\theta)}{f(\theta)}\,c(\theta)}_{\equiv\hat{\alpha}(\theta)}\quad \textit{for all } \theta\in\Theta\,. \tag{3.7}$$

The normalized deviation of the optimal action of the receiver from the middle of the interval, $\hat{\alpha}(\theta)$ defined in equation (3.7), can be interpreted as an *endogenous bias* of the receiver. It encodes the effect that the distribution of states has in communication. If the distribution is uniform, the receiver's optimal action lies in the middle of the interval, hence $\hat{\alpha}(\theta)=0$. If, for example, f is decreasing, the receiver is willing to take a relatively low action within each equilibrium interval in order to minimize her payoff loss, so in this case $\hat{\alpha}(\theta)<0$. Such an incentive is proportional to the coarseness of communication as, ceteris paribus, a bigger partition element means a higher deviation of the expected state from its center.

The receiver's deviation from the center of the interval enters both sides of equation (3.4). Proceeding as before, we can rewrite (3.6) as

$$c'(\theta)=8\,\big(b(\theta)-\hat{\alpha}(\theta)\big)\quad \text{for all } \theta\in\Theta\,. \tag{3.8}$$

Hence, the change in the coarseness of communication is proportional to the difference between the bias of the sender, $b(\theta)$, and the (endogenous) bias of the receiver, $\hat{\alpha}(\theta)$, that is, the endogenous misalignment of interest between them.

To give further intuition for equation (3.8), take two consecutive partition elements, $[\theta_{n-1},\theta_n)$ and $[\theta_n,\theta_{n+1})$. Consider first the case where $\hat{\alpha}(\theta_n)=0$ and $b(\theta_n)>0$. The sender is indifferent between the equilibrium actions $\alpha(\theta_{n-1},\theta_n)$ and $\alpha(\theta_n,\theta_{n+1})$ when the state is θ_n only if they are at the same distance from $\alpha^s(\theta_n)\equiv\theta_n+\beta\; b(\theta_n)>\theta_n$. Also, given that $\hat{\alpha}(\theta_n)=0$, the action taken by the receiver in each equilibrium interval is very close to its middle. Hence, the middle of the interval $[\theta_{n-1},\theta_n)$ is closer to θ_n than the middle of the interval $[\theta_n,\theta_{n+1})$, so $\theta_n-\theta_{n-1}<\theta_{n+1}-\theta_n$ and therefore $c'(\theta_n)>0$. Since the sender is positively biased, he is indifferent between reporting the low and the high message only if the higher equilibrium interval is bigger.

A similar argument applies in the case $\hat{\alpha}(\theta_n)<0$ and $b(\theta_n)=0$ (so f is decreasing). In this case, the equilibrium actions are at the left of the middle of the intervals. Thus, again, imposing such actions to be at the same distance of the optimal action of the sender when the state of the world is θ_n, we have $\theta_n-\theta_{n-1}<\theta_{n+1}-\theta_n$. A decreasing state distribution has the same effect than a positive bias: the sender is indifferent on sending the low or the high message only if the higher equilibrium interval is bigger.

Coarseness of communication: Equation (3.7) establishes that, since the receiver's endogenous bias only depends on the value of the primitives of the model "within" each partition element, it can be approximated by the local changes of the density function. Conversely, equation (3.8) illustrates how the change of the coarseness of communication "between" two equilibrium intervals is shaped by the sender's bias through the indifference condition (3.2). Together they determine how the coarseness of communication accumulates and de-accumulates through the state space, and can be written as

$$c'(\theta) = 8\left(b(\theta) - \frac{1}{12}\,\frac{f'(\theta)}{f(\theta)}\,c(\theta)\right) \text{ for all } \theta \in \Theta\,. \tag{3.9}$$

Any solution of (3.9) is characterized by its value at $\theta = 0$, denoted c_0, as follows:

$$c(\theta; c_0) = \frac{f(0)^{2/3}}{f(\theta)^{2/3}}\,c_0 + \frac{\int_0^\theta 8\,b(\theta')\,f(\theta')^{2/3}\,\mathrm{d}\theta'}{f(\theta)^{2/3}} \text{ for all } \theta \in \Theta\,. \tag{3.10}$$

It is clear that there is some $\underline{c}_0$ such that, $c(\theta; c_0) \geq 0$ for all θ if and only if $c_0 \geq \underline{c}_0$.

Communicable state: Equation (3.9) illustrates how the coarseness of communication changes across states as a function of the primitives. Indeed, take an increasing sequence $(\theta_n)_{n=0}^N$ satisfying the incentive equations (3.1) and (3.2). Then, if β is small enough, $c(\theta_n; \beta^{-1}\,\theta_1^2)$ approximates the $\beta^{-1}\,(\theta_n - \theta_{n-1})^2$ for all $n = 1, \ldots, N$. (Note that, since $\theta_0 = 0$, equation (3.3) establishes that $c(0)$ is approximately equal to $\beta^{-1}\,\theta_1^2$.) The proof of Proposition 3.1 shows that, in fact, for any $c_0 \geq \underline{c}_0$, if β is small enough there is some (not necessarily efficient) equilibrium where the squared size of the interval containing θ is approximately equal to β multiplied by $c(\theta)$.[9]

Given our focus on efficient equilibria, the coarseness of communication corresponds to the smallest non-negative solution of equation (3.9), that is, $c(\cdot; \underline{c}_0)$. Such solution can be written as in expression (3.3) for some (generically unique) state θ^*, called the *communicable state*, which satisfies $c(\theta^*; \underline{c}_0) = 0$.

There are only few states with the potential of being the communicable state. Given that $c(\theta^*) = 0$ and that $c(\theta) \geq 0$ for all θ, the communicable state is a minimum of $c(\cdot)$. From equation (3.8) and the fact that $\hat{\alpha}(\theta) = 0$ whenever $c(\theta) = 0$, we have that the communicable state is either 0 (only possible if $b(0) \geq 0$), or 1 (only possible if $b(1) \leq 0$), or it is an *agreement state* (as defined in Gordon, 2010); that is, satisfies $b(\theta^*) = 0$. Note that, in the third case, the bias must be locally *outward* (also as defined in Gordon, 2010) with respect

[9] Formally, for any non-negative solution $c(\cdot; c_0)$ of equation (3.9) and sequence $(\beta^i)_i \searrow 0$, there is corresponding sequence of equilibria (indexed by i) and $\theta \in \Theta$ such that $\lim_{i \to \infty} \left((\overline{\tau}^i(\theta) - \underline{\tau}^i(\theta))^2/\beta^i - c(\theta; c_0)\right) = 0$.

to the agreement state; that is, the sender prefers actions which are further away from the agreement action $a = \theta^*$ than the receiver.[10] Intuitively, since θ^* is a minimum of $c(\cdot)$, the communication is less precise in states further away from θ^*. Such decrease in the communication precision compensates the sender's incentive to deviate toward messages inducing actions further away from the agreement action $a = \theta^*$. On the contrary, when the bias is inward with respect to an agreement state (i.e., the sender prefers actions closer to the agreement state), such agreement state is never a communicable state. Hence, for example, if there is a unique agreement state $\hat{\theta}$, then $\theta^* = \hat{\theta}$ if and only if and the bias is outward with respect to $\hat{\theta}$.

The examples in Section 3.4 illustrate the location of the communicable state for linear biases. Section 4 illustrates that the location of the communicable state plays a very important role on determining the properties of equilibrium communication.

3.3 Equilibrium payoffs

The payoffs of the sender and the receiver are obtained by computing the expected payoff loss they suffer across the state space. We define the normalized payoffs of the sender and the receiver in an equilibrium as follows:

$$U^s \equiv -\beta^{-1} \int_0^1 \big(\alpha(\theta) - \theta - \beta\, b(\theta)\big)^2 f(\theta)\, \mathrm{d}\theta \quad \text{and} \quad U^r \equiv -\beta^{-1} \int_0^1 (\alpha(\theta) - \theta)^2 f(\theta)\, \mathrm{d}\theta\ .$$

Observe that the difference between the ideal actions of the sender and the receiver is $O(\beta)$, as well as the deviation of the equilibrium action from the center of each partition element (recall equation (3.7)). Also, the typical length of a partition element is, instead, $O(\beta^{1/2})$ (recall equation (3.3)). So, conditional on the state belonging to a given partition element, the expected payoff of *both* the sender and the receiver is equal to the variance of a uniform distribution in $[0,1]$ (equal to $\frac{1}{12}$) multiplied by the squared length of the interval, that is, it can be approximated by $-\frac{1}{12}\ \beta\ c(\theta)$. The previous observations are formalized in the following result about the sender's and receiver's equilibrium payoffs:

Proposition 3.3. *The equilibrium payoffs of both the sender and the receiver in an efficient equilibrium satisfy*

$$U^s \approx U^r \approx -\int_0^1 \frac{c(\theta)}{12}\, f(\theta)\, \mathrm{d}\theta\ . \tag{3.11}$$

[10] Formally, the bias is locally outward with respect to θ^* if there is a neighborhood $(\underline{\theta}, \overline{\theta}) \ni \theta^*$ such that $b(\theta) < 0$ for $\theta \in (\underline{\theta}, \theta^*)$ and $b(\theta) > 0$ for $\theta \in (\theta^*, \overline{\theta})$. The definition of (locally) *inward* is analogous.

Proposition 3.3 shows that the contribution of a state θ to the payoff of both the sender and the receiver is approximately the same, and is given by the multiplication of two terms: $\frac{1}{12}\,c(\theta)$ measures the coarseness of communication transmission of state θ, and $f(\theta)$ measures how likely the state is (see Appendix B for its generalization for arbitrary payoff functions). Hence, both the sender and the receiver benefit from better communication (lower c), especially when communication is more precise in likely states.[11]

3.4 Examples

This section provides three examples to illustrate how the coarseness of communication depends on the shape of the bias and the distribution of states.

Example 3.1 (Uniform distribution and constant bias). In order to gather some intuition on Proposition 3.1, consider the standard case where the bias is constant, and states of the world are uniformly distributed. This example is solved in Section 4 of Crawford and Sobel (1982). For the sake of clarity, we fix $b(\cdot) \equiv 1$ and focus on the sequence $(\beta^i)_i$ satisfying $\beta^i = \frac{1}{2i^2}$ for each $i \in \mathbb{N}$. In this instance, the thresholds of the equilibrium intervals in the unique efficient equilibrium are given by

$$\theta_n^i = \frac{n^2}{i^2} = 2\,n^2\,\beta^i \quad \text{for } n = 0, \ldots, i.$$

Notice that θ_n^i is a quadratic function of n, so the increments $\theta_{n+1}^i - \theta_n^i$ are linear in n or, equivalently, linear in $(\theta_n^i)^{1/2}$. As a result, in the efficient equilibrium, the squared length of a partition element is given by

$$\frac{(\theta_{n+1}^i - \theta_n^i)^2}{\beta^i} = 8\,\theta_n^i + \underbrace{8\,(\beta^i)^{1/2}\,\sqrt{2\,\theta_n^i} + 4\,\beta^i}_{\to 0 \text{ as } i \to \infty}\,. \tag{3.12}$$

In this case, $\theta^* = 0$ and $c(\theta) = 8\,\theta$ for all $\theta \in [0,1]$, so equation (3.3) holds.

[11]Under the (non-equilibrium) communication strategy $\alpha(\theta) = \theta$, the payoff loss of the sender is $O(\beta^{3/2})$ (note that equation (3.11) implies that it is $O(\beta)$ for efficient equilibrium communication), while the payoff loss of the receiver is 0. Hence, Proposition 3.3 generalizes the findings of Ottaviani (2000) and Dessein (2002) (that the receiver prefers delegation to communication when the bias is small) to a general set of payoff functions for the sender and the receiver (our result also applies for general loss functions, see Appendix B). We stress that this insight also applies to the sender: when the bias is small, he obtains an equilibrium payoff lower than if the receiver was fully informed about the state of the world and she took her preferred action.

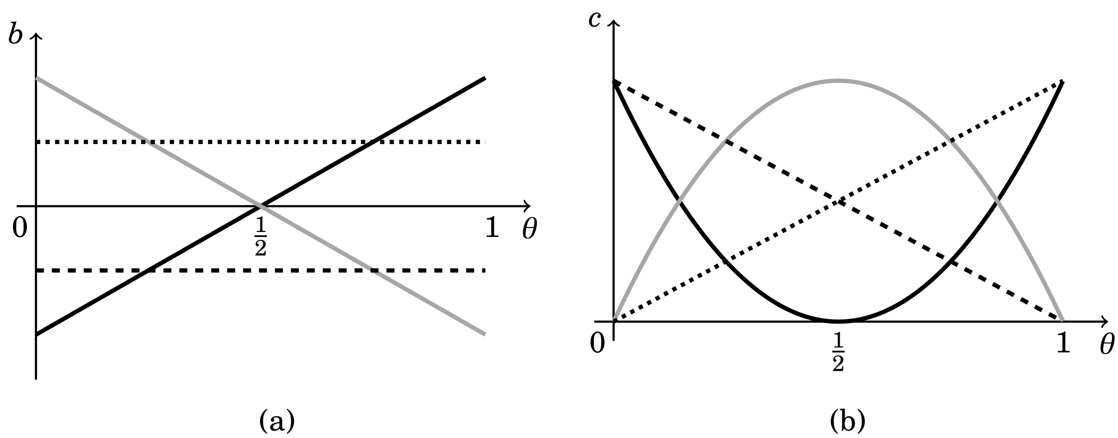


Figure 1: (a) depicts different linear biases, while in (b) we have the corresponding coarseness of communication for a uniform distribution.

Example 3.2 (Uniform distribution and linear bias). Figure 1 illustrates how the shape of the bias determines communication by providing some examples of linear biases.When the bias is constant, the coarseness of communication increases in the direction of the bias (dotted and dashed lines). When the bias is increasing (black line), the sender has the incentive to exaggerate the states towards extreme states. This implies that equilibrium partition elements need to be bigger in more extreme states while intermediate states can be communicated very precisely. The gray lines of the figure show that when, instead, the bias is decreasing with a zero at $\frac{1}{2}$, $c(\cdot)$ is increasing for low states, while it is decreasing when the state is above $\frac{1}{2}$, so information is more coarsely transmitted in intermediate states. In this case, the coarseness of communication reaches its maximum when the bias is smallest (in absolute value), that is, when $\theta = \frac{1}{2}$.[12]

Example 3.3 (Constant bias and linear distribution). We finally consider the case of a constant bias and a probability density of the form $f_\lambda(\theta) \equiv 2\lambda + 2\ (1-2\lambda)\ \theta$ for all $\theta \in \Theta$, for some $\lambda \in [0,1]$. Figure 2 depicts f_λ for some values of λ and the corresponding coarseness of communication c_λ, for a constant bias $b = 1$ (and so $\theta^* = 0$).[13] When $\lambda = \frac{1}{2}$

[12]Note that in this case there are two communicable states, 0 and 1. As Proposition 3.1 states, this is a non-generic case. Indeed, the fact that $c(0) = c(1) = 0$ is a consequence of the (anti)symmetry of the bias with respect to the middle of the state space. Kawamura (2015) makes similar observations as our example (for "overconfident" and "underconfident" senders instead of "outward" and "inward" biases).

[13]Note that this specification does not satisfy our assumptions; namely, that f_λ is not strictly positive when $\lambda \in \{0,1\}$. Nevertheless, it is not difficult to see that, as λ tends to 0 or 1, the corresponding coarseness of communication converges to the illustrated value.

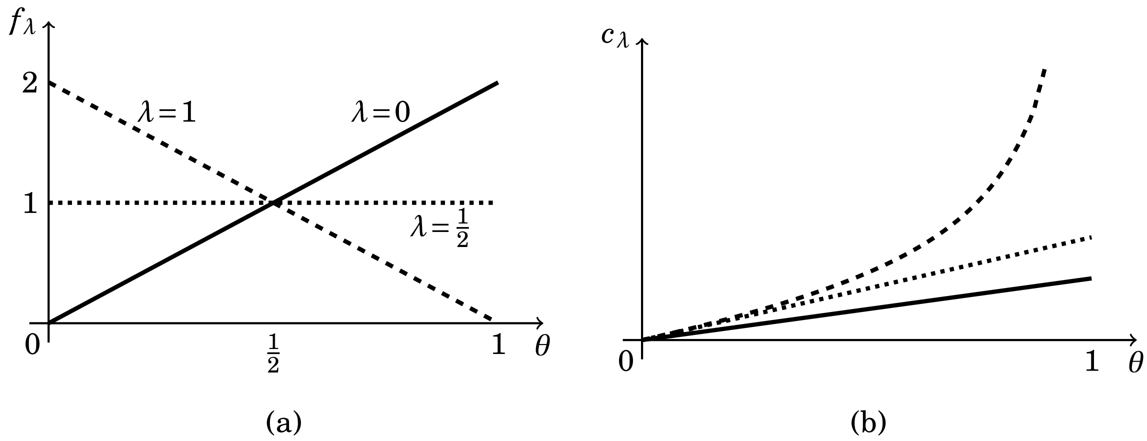


Figure 2: (a) depicts f_λ for different values of λ (see Example 3.3), while in (b) we have the corresponding coarseness of communication for $b = 1$.

the distribution of states is uniform, and so Example 3.2 applies. In this case, $c(\theta) = 8\ \theta$. When $\lambda = 0$, the distribution function is increasing, and $\hat{\alpha}(\cdot)$ is positive as a result. Hence, the sender's bias and the endogenous bias of the receiver have the same sign. Such endogenous objective alignment improves communication with respect to the case $\lambda = \frac{1}{2}$. Finally, when $\lambda = 1$, the density function is decreasing, and so $\hat{\alpha}(\cdot)$ is negative. In this case, the decreasing distribution increases the difference between the sender's and receiver's biases, hence making equilibrium communication worse than for the previous cases.

4 Comparative statics

4.1 Lowering the objective misalignment

In strategic communication models, the size of the bias is typically perceived to be the main source of the inefficiency of the equilibrium information transmission between the sender and the receiver. This common view, which is right when the bias is constant, suggests that if the conflict of interest between the sender and the receiver decreases, there are equilibria where both of them are better off.[14] This section analyzes how this view extends to a non-constant bias and obtains that, in some cases, a lower incentive alignment may lead to worse communication.

To study the effect of reducing the conflict of interest, we define a reduction of the

[14] In the uniform-quadratic-constant case, for example, increasing the absolute value of the bias reduces the maximum number of messages used in equilibrium, and makes communication less precise.

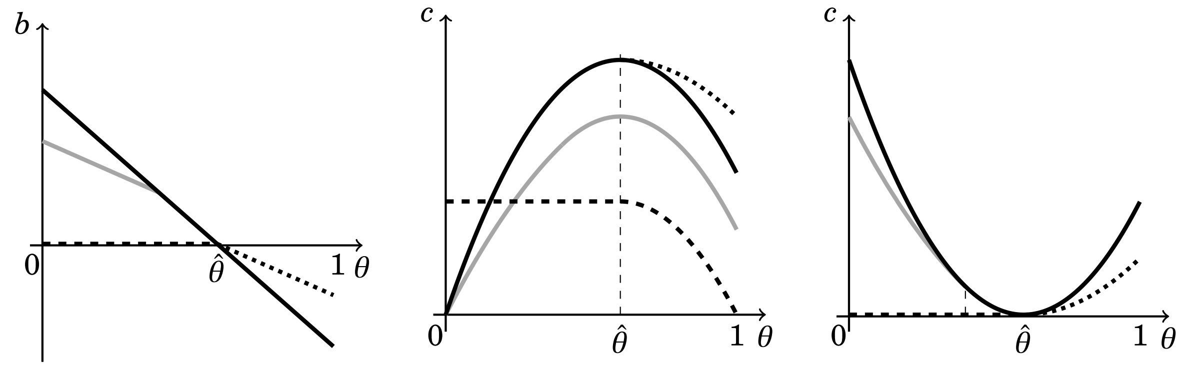
bias as follows. A *reduction of the bias b* in a set $\Theta' \subset [0,1]$ is a bias function $\hat{b}$ such that $\hat{b}(\theta) = b(\theta)$ for all $\theta \notin \Theta'$ and $\hat{b}(\theta) = \nu(\theta)\, b(\theta)$ for some $\nu(\theta) \in (0,1]$ for all $\theta \in \Theta'$. Notice that a reduction of the bias (weakly) lowers the objective misalignment between the sender and the receiver, as it makes their (state-dependent) ideal actions closer. Notice also that a reduction of the bias does not change the sign of the bias, but only the size of the incentive misalignment between the sender and the receiver. The following result characterizes the effect of a reduction of the bias:

Proposition 4.1. *Consider a reduction of the bias in a set $\Theta' \subset [0,1]$ which does not change the communicable state θ^*. Then, if the bias is inward with respect to θ^* in Θ', the coarseness of communication weakly increases in all states of the world* $[0,1]$. *The opposite is true if the bias is outward with respect to θ^* in Θ'.*

We begin with two observations that shed light on Proposition 4.1. The first observation is that if the communicable state is unique (which happens generically), small bias reductions leave the communicable state the same. In fact, it is not difficult to see that, if b is weakly increasing, then (not necessarily small) bias reductions never change the communicable state, while when b is decreasing they may change the communicable state from one extreme to the other. The second observation is that, by the expression for $c(\cdot)$ in equation (3.3), the coarseness of communication in a given state of the world θ is determined by the shape of the bias between the state and the communicable state. As a result, a local change in the value of the bias around a state θ only affects the coarseness of communication of the states further away from the communicable state. This implies that, for example, if there is a reduction of the bias in some set $[0,\theta] \subset [0,\theta^*]$ which keeps θ^* the same, such a reduction does not affect the coarseness of communication of states in $[\theta,1]$. The position of θ^* is then crucial to determine how equilibrium communication changes when the bias function changes.[15]

Proposition 4.1 is particularly relevant when the sender has an inward bias with respect to some state $\hat{\theta}$. This is perceived to be the case, for example, for the communication from central bankers to the private sector. The state θ can be a short-term optimal policy given the state of the economy, such as the optimal investment level, or interest/exchange

[15]Note that when the bias is single-signed then it is always outward with respect to the communicable state. In this case, Proposition 4.1 establishes that reductions of the bias always improve communication. This result is consistent with the findings in Chen and Gordon (2015), who analyze a version of the CS model where the sender and the receiver hold different prior beliefs. They show that when the objective misalignment decreases (they use "nestedness") the receiver's expected equilibrium payoff increases.



(a) (b) (c)

Figure 3: Illustration of Proposition 4.1. Subfigure (a) plots four biases, all inward with respect to $\frac{3}{5}$, where the gray, dashed and dotted line are biases resulting from local reductions of the black line. In (b) we depict the corresponding coarseness of communication. In (c) we depict the coarseness of communication corresponding to switching the roles of the agents, that is, multiplying by -1 all the biases.

rate, while $\hat{\theta}$ may be the long-term goal. The action a corresponds to the response of the private sector to the announcement (e.g., investment). While, under an adequate normalization, the private sector may prefer investments that match θ, the forward-looking central bank prefers a combination between the short-term and long-term efficient policies. Hence, the bias of the sender is "inward" toward the state $\hat{\theta}$.[16]

For a bias inward with respect to $\hat{\theta}$, communication is most precise at the extremes. Indeed, the incentive of the central banker to lie towards $\hat{\theta}$ is prevented, in equilibrium, by making communication of middle states less precise. Figure 3 provides an example of the bias and the coarseness of communication (black solid lines in subfigures (a) and (b)).

Assume now the central banker is replaced. The new central banker is equally concerned about low states (e.g., concerned about depressions or high inflation), but more

[16]See Blinder, Ehrmann, Fratzscher, De Haan, and Jansen (2008) for an extensive survey of central bank communication. They explain that "as it became increasingly clear that managing expectations is a central part of monetary policy, communication policy has risen in stature from a nuisance to a key instrument in the central banker's toolkit", and acknowledge that moderating or stabilizing the market's behavior is one of the implicit, and sometimes explicit, goals of central bankers. Knütter, Mohr, and Wagner (2011) and Born, Ehrmann, and Fratzscher (2014) provide empirical evidence that central bank's communication has important effects on stock prices and financial stability. See Stein (1989) and Moscarini (2007) for cheap-talk models of central banks' communication where the central bank, instead of aiming at stabilizing the exchange rate or inflation and has, as a result, the incentive to downplay its private information.

lenient on high states. This corresponds to the dotted lines in Figure 3 (a) and (b). It is then intuitive that the communication of states in $[0,\hat{\theta}]$ remains the same: the incentive compatibility constraints do not change in this region. However, this implies that higher states are communicated less precisely. The reason is the following. If the precision of communication for higher states were to remain the same, the new central banker would have the incentive to over-state the value of θ.[17] In the new equilibrium, the incentives of the central banker are kept by slowing the rate at which the precision communication increases in the upper region of the state space.

Figure 3 (b) illustrates that communication improves for all states if the new central banker is instead more lenient in low states (gray lines). Now, the incentive constraints are relaxed for lower states, which implies that middle states are communicated more precisely, and the communication of high states is also more precise as a result. If we keep shrinking the central banker's bias in lower states, there is a moment where the communication of the highest states becomes very precise. Formally, the communicable state jumps from 0 to 1 at this point. After this point, the logic described above applies for lower states: making the central banker more lenient on lower states worsens communication. Even if the bias is very small for lower states (dashed lines), their equilibrium communication needs to be imprecise to keep the incentive compatibility constraints of the central banker at middle and high states.

When, on the contrary, the sender has an outward bias (with respect to a state $\hat{\theta}$), reducing the size of the bias always improves communication. Indeed, in this case, the communicable state coincides with $\hat{\theta}$, which implies that the coarseness of communication increases toward extreme states (see equation (3.3)). As a result, any reduction of the bias lowers the distortion that extreme states generate, which results in an improvement of the equilibrium communication. By Proposition 3.3, such improvement of the equilibrium communication increases the payoffs of both the sender and the receiver.

Figure 3 (c) shows the effect of the previous reductions of the bias if all the biases in Figure 3 (a) are multiplied by -1, which is equivalent to assume that the sender and the receiver swap their roles. In this case the bias is outward with respect to $\hat{\theta}$, which now coincides with the communicable state. By Proposition 4.1, all local reductions of the bias

[17] Intuitively, for two consecutive equilibrium intervals of high states, the action corresponding to the higher interval is closer to the threshold state than the action corresponding to the lower interval, because the bias is negative on $[\hat{\theta},1]$. If the sender becomes less biased on $[\hat{\theta},1]$, the threshold type becomes more willing to take the upper action, hence the upper interval becomes larger in equilibrium.

improve communication. Notice that, when the reduction is extreme in one of the sides of the communicable state (dashed line), communication becomes very precise in this side of the state space.

The effect of a reduction of the bias affecting both inward and outward states with respect to θ^* is, in general, ambiguous. In the case of a uniform reduction of the bias–i.e., multiplying the bias in all states by the same multiplying factor in $(0,1)$–, the effect is nevertheless clear: it decreases the coarseness of communication in all states by the same multiplying factor. There is then a sense in which the conventional wisdom that a higher objective alignment improves communication.

4.2 Role assignment for better communication

In this section we study situations where the assignment of the roles of sender and receiver to agents is a choice variable of a third party. Re-assigning roles changes the sign of the relative bias of the sender and the receiver, and therefore the communicable state and equilibrium information transmission. We ask which role assignment minimizes the payoff loss from communication (recall that, by equation (3.11), the sender's and receiver's payoffs are approximately proportional to the negative of the expected coarseness of communication).

Proposition 4.2. *Let f be uniform and let $c(\cdot;b)$ denote the coarseness of communication for each bias shape b. Then, if b is increasing,* $\mathbb{E}[c(\tilde{\theta};b)] \leq \mathbb{E}[c(\tilde{\theta};-b)]$.

Proposition 4.2 may be surprising at first sight, since both the state space and the distribution function are symmetric around state $\frac{1}{2}$. Nevertheless, note that the monotonicity of b determines whether the bias points toward the states with more agreement or not, and hence is preserved under reversals of the state space.[18] When b is increasing, the communicable state is equal to 0 if b is always positive, equal to the agreement state if b has a zero, and equal to 1 if b is always negative (see, for example, Figure 1). Hence, if b is increasing, the bias is outward with respect to the communicable state, which coincides a the state with highest objective agreement (i.e., θ^* is a minimum of $|b(\cdot)|$). When b is decreasing, instead, the communicable state is equal to 1 if b is always positive, equal to either 0 or 1 if b has a zero, and equal to 0 if b is always negative. A decreasing bias is then inward with respect to the communicable state.

[18]Indeed, reversing the state space through the transformation $\rho(\theta) = 1-\theta$ implies that the new bias at θ is $1-\theta-\beta\, b(1-\theta)-(1-\theta) = -\beta\, b(1-\theta)$. Then, if b is strictly increasing, so is $-b\circ\rho$, and $c(\theta;b)=c(\rho(\theta);-b\circ\rho)$.

To provide an intuition for Proposition 4.2, consider the following example. A firm has to assign two workers (managers) to different tasks (divisions) for a new project. One task (sender) involves obtaining and transmitting information about the output-maximizing effort level, θ, while the other task (receiver) requires acquiring the expertise of how to implement the joint effort, and deciding its level a. Absent of effort costs, both workers want the effort to match the state. Nevertheless, we assume that each worker $i \in \{1,2\}$ incurs a small quadratic cost of effort $k_i\, a^2$, where $k_i \geq 0$. We assume that $k_1 < k_2$, so agent 1 is more efficient than agent 2. Their utility functions are:

$$u_i(\theta,a) \;=\; -(a-\theta)^2 - k_i\, a^2 \;=\; -\tfrac{k_i}{1+k_i}\,\theta^2 - (1+k_i)\left(a - \tfrac{1}{1+k_i}\,\theta\right)^2 .$$

If the more efficient worker is assigned to be the sender, the bias is positive and increasing (and equal to $\left(\frac{1}{1+k_1} - \frac{1}{1+k_2}\right)\theta$), hence the communicable state is $\theta^* = 0$. Low states are communicated precisely and the incentive to over-report the state is prevented, in equilibrium, with a coarser communication of high states. Importantly, the increase in the coarseness of communication is slow in low states, since the bias is small in this region. If, instead, the less efficient worker is the sender, the bias is multiplied by -1, and the communicable state becomes $\theta^* = 1$. In this case, the incentive to under-report implies that low states are communicated less precisely. Crucially, now the increase of the coarseness toward low states is fast in the states which are communicated more precisely.[19] As a result, the overall coarseness of communication is higher in the second case, and hence both agents are better off when the more efficient agent is assigned to be the sender.

The previous example can be generalized to the case where workers differ on the cost of choosing a state different from some state $\hat{\theta}_i \in \Theta$ (in the previous example, $\hat{\theta}_1 = \hat{\theta}_2 = 0$):

$$u_i(\theta,a) \;=\; -(a-\theta)^2 - k_i\,(a-\hat{\theta}_i)^2 .$$

The state $\hat{\theta}_i$ could be interpreted as a long-term idiosyncratic goal, while $k_i \geq 0$ captures the willingness to compromise long-term versus short-term optimal investments.[20] In

[19] Intuitively, when the sender is the efficient worker, the "most disruptive" states (the ones with the highest objective misalignment) are at the opposite extreme of the state space than the communicable state, and therefore the distortion they generate in communication is minimal. The opposite occurs when the inefficient worker is the sender: in this case, the bias is maximal at the communicable state.

[20] The differences in $\hat{\theta}$'s and k's could follow from the workers being at different stages of their career. For example, Ottaviani and Sørensen (2006) show that, in a contest (winner takes all) theory of forecasting, forecasters may have the incentive to exaggerate their information, while in a reputational theory of forecasting, they have the incentive to shade it towards the prior mean.

this case, it is easy to see that assigning the less foresighted agent (with smaller k_i) to be the sender improves communication, irrespective of the value of $\hat{\theta}_1$ and $\hat{\theta}_2$. The bias is small at the communicable state under this role assignment, so the overall coarseness of communication increases slower than when the roles are reversed.

5 Conclusions

Our paper shows that studying communication under the assumption that the bias is small preserves the fundamental tradeoffs that strategic communication transmission involves while significantly adding tractability to the analysis. We shed light on the subtleties that studying communication implies.

When the bias is not big, neighboring states should be communicated with similar precision since, otherwise, the sender has the incentive to deviate when the realized state is supposed to be communicated imprecisely. Similarly, the incentive to over- or under-report the state is prevented, in equilibrium, by an increasing or decreasing imprecision of the language, respectively. These two intuitive rules sometimes interact in non-trivial ways. In central bank communication, a smaller bias in some states may make the information transmission more coarse. In joint projects, assigning more efficient or less forward-looking workers to undertake tasks involving communication may improve the project's output.

The application of the new tools and results to other cheap-talk settings, such as communication with multiple senders or dynamic information transmission, is left to future research.

A Proofs

A.1 A preliminary result

We begin with a preliminary result in our base model which is going to be useful for the rest of the proofs. The result claims that, for a fixed shape of the bias b and decreasing sequence of bias sizes $(\beta^i)_i$, the size of the largest interval of a sequence of equilibria decreases at most as fast as $(\beta^i)^{1/2}$.

Lemma A.1. *Fix a sequence $(\beta^i)_i$ converging to 0 from above, and a corresponding sequence of (not-necessarily efficient) equilibria, where for each $i \in \mathbb{N}$ we use $\{\theta^i_n | n = 0, ..., N^i\}$ to denote the set interval thresholds of the i-th equilibrium (in the model with bias $\beta^i\ b(\cdot)$). Then, we have*

$$\liminf_{i\to\infty} \frac{\max_n |\theta^i_{n+1} - \theta^i_n|}{(\beta^i)^{1/2}} > 0 \ .$$

Proof. Assume, for simplicity, that $\int_0^1 |b(\theta)|\ \mathrm{d}\theta = 1$ (note that b and β can be normalized so that this condition is satisfied). Fix a sequence $(\beta^i)_i$ and a corresponding sequence of equilibria. Fix a state $\bar{\theta} \in (0,1)$ such that $|b(\bar{\theta})| > 1/2$. Assume, for simplicity, that $\bar{\theta}$ can be chosen so that $b(\bar{\theta}) > 0$ (otherwise the argument is analogous). It is convenient to use $[\theta^i_0, \theta^i_1)$ to denote the equilibrium interval containing $\bar{\theta}$ in the i-th equilibrium, while allowing the index a partition element to be negative, so the i-the equilibrium thresholds are $\{\theta^i_n | n = -N^i_-, ..., N^i_+\}$.

Step 1: We first prove that $\liminf_{i\to\infty} \frac{\max_n (\theta^i_{n+1} - \theta^i_n)^2}{(\beta^i)^2} > 0$. By the indifference condition of the sender (equation (3.2)) and the symmetry of his payoff function we have, for all $n = N^i_- + 1, ..., N^i_+ - 1$,

$$\theta^i_n + \beta^i\ b(\theta^i_n) - \alpha(\theta^i_{n-1}, \theta^i_n) = \alpha(\theta^i_n, \theta^i_{n+1}) - \theta^i_n - \beta^i\ b(\theta^i_n) \ ,$$

where α is defined in equation (3.1). Since $\alpha(\theta^i_{n-1}, \theta^i_n) \in (\theta^i_{n-1}, \theta^i_n)$ and $\alpha(\theta^i_n, \theta^i_{n+1}) \in (\theta^i_n, \theta^i_{n+1})$ for all n,[21] we have that

$$\theta^i_1 - \theta^i_0 > \alpha(\theta^i_0, \theta^i_1) - \theta^i_0 = \theta^i_0 - \alpha(\theta^i_{-1}, \theta^i_0) + 2\ \beta^i\ b(\theta^i_0) \geq 2\ \beta^i\ b(\theta^i_0) > \beta^i \ .$$

So, it follows that $\liminf_{i\to\infty} \frac{\max_n (\theta^i_{n+1} - \theta^i_n)^2}{(\beta^i)^2} \geq \liminf_{i\to\infty} \frac{(\theta^i_1 - \theta^i_0)^2}{(\beta^i)^2} \geq 1 > 0$.

Step 2: We now assume that $\liminf_{i\to\infty} \frac{\max_n (\theta^i_{n+1} - \theta^i_n)^2}{(\beta^i)^2} = q^2$, for some $q \in \mathbb{R}_{++}$. Fix $Q \equiv \lceil q \rceil \in \mathbb{N}$. Notice that, since $\bar{\theta} \in (0,1)$, Q is independent of i. For this reason, if i is large enough, $\theta^i_{-Q} > 0$. Using the sender's indifference condition at the bounds of the equilibrium intervals (equation (3.2)) and the fact that, for all n,

$$\alpha(\theta^i_n, \theta^i_{n+1}) = \frac{\theta^i_n + \theta^i_{n+1}}{2} + O((\theta^i_{n+1} - \theta^i_n)^2)$$

[21]This is a consequence of the single peakedness of u^r and the fact that, for each given state θ, the action which minimizes the payoff loss of the receiver is $a = \theta$.

we obtain:

$$\alpha(\theta^i_{-Q},\theta^i_{-Q+1})+\theta^i_{-Q+1}+\beta^i\ b(\theta^i_{-Q+1})=\theta^i_{-Q+1}+\beta^i\ b(\theta^i_{-Q+1})-\alpha(\theta^i_{-Q+1},\theta^i_{-Q+2})$$
$$\Rightarrow\ \theta^i_{-Q+1}-\theta^i_{-Q}=\theta^i_{-Q+2}-\theta^i_{-Q+1}-4\ \beta^i\ b(\theta^i_{-Q+1})+o(\beta^i)\ .$$

As i increases, since b is Lipchitz continuous with a Lipchitz constant independent of i, we have that $\beta^i\ b(\theta^i_n)=\beta^i\ b(\theta^i_0)+o(\beta^i)$ for all $n\in\{-Q,...,0\}$. Thus, recursively using the previous formula Q times, we obtain

$$\begin{aligned}\theta^i_{-Q+1}-\theta^i_{-Q} &= \theta^i_1-\theta^i_0-4\overbrace{\sum_{n=-Q+1}^{0}\beta^i\ b(\theta^i_n)}^{=Q\ \beta^i\ b(\theta^i_0)+o(\beta^i)<Q\ \beta^i/2+o(\beta^i)}+o(\beta^i)\\ &< \theta^i_1-\theta^i_0-Q\ 2\ \beta^i+o(\beta^i)\ <\ 0\ .\end{aligned}$$

This is a clear contradiction.

Step 3: Assume finally $\liminf_{i\to\infty}\frac{\max_n(\theta^i_{n+1}-\theta^i_n)^2}{\beta^i}=0$ but $\liminf_{i\to\infty}\frac{\max_n(\theta^i_{n+1}-\theta^i_n)^2}{(\beta^i)^2}=+\infty$. Let us define, for each $i\in\mathbb{N}$, $Q^i\equiv\lceil(\beta^i)^{-1/2}\rceil$. Since, by assumption, $\theta^i_1-\theta^i_0=o((\beta^i)^{1/2})$, we have that $\lim_{i\to\infty}\theta^i_0-\theta^i_{-Q^i}=0$, so in particular $\beta\ b^i(\theta^i_n)>\beta^i/4$ for all $n=-Q^i,...,0$ if i is high enough. Using again the indifference condition for the receiver we now have that, as $i\to\infty$,

$$\begin{aligned}\theta^i_{-Q^i+1}-\theta^i_{-Q^i} &= \theta^i_1-\theta^i_0-\sum_{n=1}^{Q^i}4\ \beta^i\ b(\theta^i_{-n})+o(\beta^i)\\ &< \theta^i_1-\theta^i_0-\frac{1}{(\beta^i)^{1/2}}\ 4\ \frac{\beta^i}{4}+o((\beta^i)^{1/2})=\theta^i_1-\theta^i_0-(\beta^i)^{1/2}+o((\beta^i)^{1/2})\ .\end{aligned}$$

This is a contradiction again since, when i is high enough, the right hand side of the previous expression is negative. □

A.2 Proofs of the results in Section 3

Proof of Propositions 3.1-3.3

Proof. We fix a sequence $(\beta^i)_i$ strictly decreasing to 0 and a bias function b. It is convenient to define $\Delta^i\equiv(\beta^i)^{1/2}$ for all i.

Lemma A.1 shows that, as i increases, the length of the longest equilibrium interval of an equilibrium decreases at most as fast as Δ^i. In this proof, we show that for any sequence of efficient equilibria such a length decreases exactly as Δ^i, and we use it to prove statements of Propositions 3.1-3.3 and B.2 (steps 5 to 8).

Step 1: The difference equation. Fix some $\gamma_0\in\mathbb{R}_{++}$. We now fix some i, and the corresponding Δ^i. Then, we define $\theta^i_0=0$ and $\gamma^i_0=\gamma_0$ and, for all $n\in\mathbb{N}$, we define θ^i_n and γ^i_n iteratively from θ^i_{n-1} and γ^i_{n-1} as follows:

1. $\theta^i_n \equiv \theta^i_{n-1} + \Delta^i\ \gamma^i_{n-1}$.

2. Using the previous equation for θ^i_n, γ^i_n is obtained solving

$$\theta^i_n + (\Delta^i)^2\ b(\theta^i_n) - \alpha(\theta^i_{n-1}, \theta^i_n) = \alpha(\theta^i_n, \theta^i_n + \Delta^i\ \gamma^i_n) - (\theta^i_n + (\Delta^i)^2\ b(\theta^i_n)) \tag{A.1}$$

when it exists, where α is defined in equation (3.1). Notice that for any given θ^i_{n-1} and θ^i_n, if γ^i_n solving (A.1) exists, it is uniquely defined, since $\alpha(\theta^i_n, \cdot)$ is strictly increasing. A value γ^i_n solving (A.1) does not exist only if one of the following two cases holds. The first case is when the right hand side of the equation is bigger than the left hand side even when $\theta^i_n + \Delta^i\ \gamma^i_n = 0$: in this case we set $\gamma^i_n = -\theta^i_n/\Delta^i$, so $\theta^i_{n'} = 0$ and $\gamma^i_{n'} = 0$ for all $n' > n$. The second case is when if the right hand side is smaller than the left hand side even when $\theta^i_n + \Delta^i\ \gamma^i_n = 1$: in this case we set $\gamma^i_n = \frac{1-\theta^i_n}{\Delta^i}$, so $\theta^i_{n'} = 1$ and $\gamma^i_{n'} = 0$ for all $n' > n$.

For each pair $(\theta, \gamma) \in \Theta \times \mathbb{R}$, let $g^i(\theta, \gamma)$ be such that equation (A.1) holds when θ^i_{n-1}, θ^i_n and γ^i_n are replaced, respectively, by θ, $\theta + \Delta^i\ \gamma$ and $\gamma + \Delta^i\ g^i(\theta, \gamma)$ (with the values described above when no solution of (A.1) exists). Note that, while $\gamma^i_n \equiv \frac{\theta^i_n - \theta^i_{n-1}}{\Delta^i}$ is indicative of how fast θ changes, $g(\theta^i_{n-1}, \gamma^i_{n-1}) \equiv \frac{\gamma^i_n - \gamma^i_{n-1}}{\Delta^i}$ is indicative of how fast γ changes.

The first-order condition for the receiver's optimality (equation (3.1)) is

$$0 = \int_{\theta^i_n}^{\theta^i_{n+1}} \frac{f(\tilde{\theta})}{F(\theta_{n+1}) - F(\theta_n)}\ 2\ (\tilde{\theta} - \alpha(\theta^i_n, \theta^i_{n+1}))\ \mathrm{d}\tilde{\theta}\ .$$

The previous integral can be approximated using a first-order approximation of the distribution function. Hence, replacing $f(\tilde{\theta})$ by $f(\theta_n) + (\tilde{\theta} - \theta_n)\ f'(\theta_n) + O((\tilde{\theta} - \theta_n)^2)$, we obtain

$$\alpha(\theta^i_n, \theta^i_{n+1}) - \frac{\theta^i_n + \theta^i_{n+1}}{2} = \frac{1}{12} \frac{f'(\theta^i_n)}{f(\theta^i_n)}\ (\theta^i_{n+1} - \theta^i_n)^2 + O((\theta^i_{n+1} - \theta^i_n)^3) \tag{A.2}$$

as $i \to \infty$. Using this approximation and equation (A.1), we can also approximate g as

$$g^i(\theta, \gamma) = 4\ b(\theta) - \frac{\gamma^2}{3}\ \frac{f'(\theta)}{f(\theta)} + O(\Delta^i)\ ,$$

which allows us to write the system of difference equations described above as

$$\theta^i_{n+1} = \theta^i_n + \Delta^i\ \gamma^i_n\ , \tag{A.3}$$

$$\gamma^i_{n+1} = \gamma^i_n + \Delta^i\ \left(4\ b(\theta^i_n) - \frac{(\gamma^i_n)^2}{3}\ \frac{f'(\theta^i_n)}{f(\theta^i_n)}\right) + o(\Delta^i)\ . \tag{A.4}$$

Step 2: Solution to the difference equation as $i \to 0$**.** We now apply standard difference equations analysis to approximate the solution to the difference equation by a differential equation.

It then follows (see Theorem 1.1 in Iserles (2009)) that there exist two functions $\tilde{\theta}(\cdot)$ and $\tilde{\gamma}(\cdot)$ such that, for all $s > 0$ we have[22]

$$\lim_{i\to\infty} \max_{s'\in[0,s]} \left(|\theta^i_{\lceil s'/\Delta^i\rceil} - \tilde{\theta}(s')| + |\gamma^i_{\lceil s'/\Delta^i\rceil} - \tilde{\gamma}(s')|\right) = 0 \ ,$$

and $\tilde{\theta}$ and $\tilde{\gamma}$ satisfy the following differential equations for all $s > 0$ such that $\tilde{\theta}(s) \in (0,1)$

$$\tilde{\theta}'(s) = \tilde{\gamma}(s) \ , \tag{A.5}$$

$$\tilde{\gamma}'(s) = 4\, b(\tilde{\theta}(s)) - \frac{\tilde{\gamma}(s)^2}{3} \frac{f'(\tilde{\theta}(s))}{f(\tilde{\theta}(s))} \ , \tag{A.6}$$

with initial conditions $\theta(0) = 0$ and $\lim_{s\searrow 0} \tilde{\theta}'(s) = \tilde{\gamma}(0) = \gamma_0$, and $\gamma(s') = 0$ for all $s' > s$ whenever $\tilde{\theta}(s) \in \{0,1\}$ for some $s > 0$. From now on, we use $(\tilde{\theta}(\cdot;\gamma_0), \tilde{\gamma}(\cdot;\gamma_0))$ to denote the solutions of (A.5)-(A.6) satisfying such boundary conditions.

Step 3: Existence of θ^* (as defined in Proposition 3.1). The following result establishes that there is some value $\bar{\gamma}_0$ such that, for all $\gamma_0 > \bar{\gamma}_0$ we have that $\tilde{\theta}(\cdot;\gamma_0)$ is strictly increasing. As indicated in the proof, this implies the existence of θ^*.

Lemma A.2. *There exists some $\bar{\gamma}_0 \geq 0$ such that: $\gamma_0 > \bar{\gamma}_0$ if and only if there exists some $s^*(\gamma_0) > 0$ satisfying $\tilde{\theta}(s^*(\gamma_0);\gamma_0) = 1$ and $\tilde{\gamma}(s;\gamma_0) > 0$ for all $s \in [0, s^*(\gamma_0)]$.*

Proof. For a fixed $\gamma_0 > 0$, let's define

$$s^*(\gamma_0) = \sup\left\{ s \,\middle|\, \tilde{\theta}(s';\gamma_0) < 1 \text{ and } \tilde{\gamma}(s';\gamma_0) > 0 \text{ for all } s' \in [0,s] \right\} \in \bar{\mathbb{R}}_{++} \ .$$

So, $\tilde{\theta}(\cdot;\gamma_0)$ is strictly increasing in $(0, s^*(\gamma_0))$. Then, we can rewrite equation (A.6) in terms of $\tilde{c}(\theta;\gamma_0) \equiv \tilde{\gamma}(\tilde{\theta}^{-1}(\theta;\gamma_0);\gamma_0)^2$, so it is easy to verify that $\tilde{c}(\cdot;\gamma_0)$ satisfies equation (3.9) on $(0, s^*(\gamma_0))$.

Note first that two of different solutions of (3.9) cannot coincide in a state since, otherwise, by the Picard-Lindelöf theorem, they would coincide everywhere. As a result, for all θ, $\tilde{c}(\theta;\gamma_0)$ is strictly increasing in γ_0. Now, for the sake of contradiction, assume that $\lim_{\gamma_0\to\infty} \tilde{c}(\theta;\gamma_0) = K < \infty$ for some $\theta \in \Theta$. Standard ODE analysis guarantees that there is a solution of the equation (3.9) satisfying $\tilde{c}(\theta;\gamma_0) = K + \varepsilon$ for all $\varepsilon > 0$, which is a contradiction. So, if γ_0 is high enough, the solution of (3.9) satisfies $\tilde{c}(\theta;\gamma_0) > 0$ for all $\theta \in \Theta$, and as a result $\tilde{\gamma}(s;\gamma_0) > 0$ for all $s \in [0, s^*(\gamma_0)]$ and $\tilde{\theta}(s^*(\gamma_0);\gamma_0) = 1$. We then have that $\bar{\gamma}_0 = \inf\{\gamma_0 > 0 | \tilde{c}(\theta;\gamma_0) > 0 \ \forall \theta \in [0,1]\}$, so we have that $\tilde{c}(\theta;\bar{\gamma}_0) \geq 0$ and $\tilde{c}(\theta^*;\bar{\gamma}_0) = 0$ for a (generically unique) state $\theta^* \in \Theta$. □

[22]Notice that the right hand side of equation (A.4) is not Lipschitz-continuous, since the partial derivative with respect to γ^i_n when it is large is unbounded. Nevertheless, one can make it Lipschitz-continuous by simply replacing "$(\gamma^i_n)^2$" by "$\max\{K, (\gamma^i_n)^2\}$ for some big $K > 0$, to then, one can apply Theorem 1.1 in Iserles (2009). Given that, in an equilibrium, c is bounded, it is easy to see that if K is big enough this has no effect on our results.

Step 4: Properties of $\bar{\gamma}_0$. We now present some properties of $\bar{\gamma}_0$. Fix some $\gamma_0 > \bar{\gamma}_0$. Let $N^i(\gamma_0)$ be the smallest so that the right hand side of equation (A.1) (for $n = N^i(\gamma_0)$) is higher than the left hand side even when $\theta_{N^i(\gamma_0)+1} = 1$. Notice that, given our previous results, $N^i(\gamma_0)$ exists if i is high enough, and $N^i(\gamma_0) = O((\Delta^i)^{-1})$. In fact, $N^i(\gamma_0)$ can be approximated as follows:

$$N^i(\gamma_0) = \sum_{n=1}^{N^i(\gamma_0)} 1 = \sum_{n=1}^{N^i(\gamma_0)} \int_{\theta^i_{n-1}}^{\theta^i_n} \frac{1}{\theta^i_n - \theta^i_{n-1}} \, \mathrm{d}\theta = \frac{1}{\Delta^i} \int_0^1 \frac{1}{\tilde{\gamma}(\theta;\gamma_0)} \, \mathrm{d}\theta + O((\Delta^i)^0) \,. \tag{A.7}$$

Note further that, if $\gamma_0 > \gamma_0' > \bar{\gamma}_0$ and i is high enough, then $N^i(\gamma_0) < N^i(\gamma_0')$.

In the system of difference equations (A.3)-(A.4), (θ^i_n, γ^i_n) are continuous functions of γ_0. As a result, for a fixed $\gamma_0 \geq \bar{\gamma}_0$, there exists a sequence $(\gamma^i_0)_i$ converging to γ_0 such that $\theta^i_{N^i(\gamma_0)} = 1$, that is, where the strictly increasing sequence $(\theta^i_n)_{n=0}^{N^i(\gamma_0)}$ forms the thresholds of the partition elements of an equilibrium for each i. This indicates that, for any $\gamma_0 \geq \bar{\gamma}_0$ and sequence $\Delta^i \searrow 0$, there exists a sequence $(\gamma^i_0)_i$ converging to γ_0 and a sequence of equilibria such that $\theta^i_1 - \theta^i_0 = \gamma^i_0 \, \Delta_i$. Furthermore, for any sequence of such equilibria, $\theta^i_{n+1} - \theta^i_n = \Delta^i \; \tilde{\gamma}(\tilde{\theta}^{-1}(\theta^i_n); \gamma^i_0) + o(\Delta^i)$ for all n. It is also clear that, if $\gamma_0 < \tilde{\gamma}_0$, there is no such a sequence, since $\tilde{\gamma}(\theta, \gamma_0) < 0$ for some $\theta \in \Theta$.

Fix $\gamma_0 \geq \bar{\gamma}_0$ and a sequence of equilibria such that $\lim_{i\to\infty} \frac{\theta^i_1 - \theta^i_0}{\Delta_i} = \gamma_0$. Recall that, as we defined in the proof of Lemma A.2, $\tilde{c}(\theta;\gamma_0) = \tilde{\gamma}(\tilde{\theta}^{-1}(\theta;\gamma_0);\gamma_0)^2$, and that $\tilde{\gamma}(\tilde{\theta}^{-1}(\theta;\gamma_0);\gamma_0)$ is increasing in γ_0 for all $\theta \in [0,1]$. Also, we have that $(\overline{\tau}^i(\theta) - \underline{\tau}^i(\theta))^2 = \tilde{c}(\theta;\gamma_0) \, (\Delta^i)^2 + o((\Delta_i)^2)$. We can then compute the payoff of the receiver as follows:

$$\begin{aligned}
\int_0^1 u^r(\theta, \alpha(\underline{\tau}^i(\theta), \overline{\tau}^i(\theta))) \, f(\theta) \, \mathrm{d}\theta &= -\int_0^1 \Big(\frac{\overline{\tau}^i(\theta) + \underline{\tau}^i(\theta)}{2} - \theta\Big)^2 f(\theta) \, \mathrm{d}\theta + o((\Delta^i)^2) \\
&= -\int_0^1 \Big(\frac{\overline{\tau}^i(\theta) + \underline{\tau}^i(\theta)}{2} - \theta\Big)^2 f(\overline{\tau}^i(\theta)) \, \mathrm{d}\theta + o((\Delta^i)^2) \\
&= -\int_0^1 \frac{\big(\overline{\tau}^i(\theta) - \underline{\tau}^i(\theta)\big)^2}{12} f(\overline{\tau}(\theta)) \, \mathrm{d}\theta + o((\Delta^i)^2) \\
&= -(\Delta^i)^2 \, \frac{1}{12} \int_0^1 c(\theta;\gamma_0) \, f(\theta) \, \mathrm{d}\theta + o((\Delta^i)^2)
\end{aligned} \tag{A.8}$$

where, for the third equality, we used that $\int_{\underline{\tau}^i(\theta)}^{\overline{\tau}^i(\theta)} \Big(\frac{\overline{\tau}(\theta) + \underline{\tau}(\theta)}{2} - \tilde{\theta}\Big)^2 \, \mathrm{d}\tilde{\theta} = \frac{1}{12} (\overline{\tau}^i(\theta) - \underline{\tau}^i(\theta))^2$. The expected payoff loss of the sender can be calculated analogously. Hence, for any two values $\gamma_0 > \gamma_0' \geq \bar{\gamma}_0$, and for two sequences of equilibria with $\gamma^i_0 = \gamma_0 + o((\Delta^i)^0)$ and $\gamma'^i_0 = \gamma_0' + o((\Delta^i)^0)$, respectively, we have that there is some $\bar{i}$ such that the i-th equilibrium the second sequence gives a higher payoff to both the sender and the receiver than the i-th equilibrium in the first sequence for all $i \geq \bar{i}$.

Step 5: Proof of Proposition B.2. The first statement in Proposition B.2 follows from Step 4. The following lemma implies the second statement of Proposition B.2:

Lemma A.3. *Fix, for each $i \in \mathbb{N}$, an equilibrium in the model with bias $b^i(\cdot) = (\Delta^i)^2 \, b(\cdot)$. Let N^i be number of messages used the i-th equilibrium, assume it is finite for each i, and that $N^i \to \infty$ as*

$i \to \infty$. *Then, if* $\lim_{i\to\infty} \Delta_i \; N^i = 0$, *then there exists some* $k' > 0$ *such that*

$$\lim_{i\to\infty} \frac{(\overline{\tau}^i(\theta) - \underline{\tau}^i(\theta))\; N^i}{f(\theta)^{1/3}} = k' \;\; \textit{for all } \theta \in \Theta. \tag{A.9}$$

In this case, the equilibrium payoff loss of both the sender and the receiver are $O((N^i)^{-2}) > O((\Delta^i)^2)$ *as* $i \to \infty$.

Proof. Fix some $\gamma_0 \in \mathbb{R}_{++}$ and some function $h : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $\lim_{x \searrow 0} h(x)/x = +\infty$. Consider, for each i, a pair sequences $(\theta^i_n)_n$ and $(\gamma^i_n)_n$ following the equation

$$\theta^i_n \equiv \theta^i_{n-1} + h(\Delta^i)\; \gamma^i_{n-1}$$

and equation (A.1). Define $\tilde{\Delta}^i \equiv h(\Delta^i)$, so note that $\Delta^i = o(\tilde{\Delta}^i)$. It is easy to see that the difference equations (A.3) and (A.4) now hold identically replacing Δ^i by $\tilde{\Delta}^i$, with the exception that the term "4 $b(\theta^i_n)$" is now equal to 0 (since this term is $O(\Delta^i)$ while the rest are $O(\tilde{\Delta}^i)$). This implies that equation (A.6) now is $\tilde{\gamma}'(s) = -\frac{\tilde{\gamma}(s)^2}{3}\,\frac{f'(\tilde{\theta}(s))}{f(\tilde{\theta}(s))}$, so $\tilde{\gamma}(s) = C_1\; f(\tilde{\theta}(s))^{1/3}$ for some $C_1 > 0$.

The number of partition elements of each equilibrium satisfies equation (A.7) replacing Δ^i by $\tilde{\Delta}^i$, so $\lim_{i\to\infty} \Delta_i \; N^i = 0$. Hence, proceeding as in the first part of the proof, it is then easy to see that equation (A.9) holds.

Finally, the equation for the payoff (A.8) is the same, but since now $c(\theta) = O((\tilde{\Delta}^i)^2)$ (instead of $O((\Delta^i)^2)$, as before) we have that the payoff is itself $O((\tilde{\Delta}^i)^2)$, that is, $O((N^i)^{-2})$. □

Step 6: Proof of Proposition 3.1. We prove that any sequence of efficient equilibria satisfy

$$\frac{\theta^i_{n+1} - \theta^i_n}{\Delta^i} \to \bar{\gamma}_0 \; . \tag{A.10}$$

The results in the previous parts of the proof imply that there are sequences of equilibria satisfying equation (A.10). Furthermore, if a sequence of equilibria is such that equation (A.10) is not satisfied (so there is a sequence of equilibria where $\lim_{i\to\infty} \frac{\theta^i_{n+1}-\theta^i_n}{\Delta^i} = \gamma_0 \in (\bar{\gamma}_0, +\infty]$), then there is a (potentially large) $\bar{i} \in \mathbb{N}$ such that the payoff of the $\bar{i}$-th equilibrium is strictly lower than the payoff of the $\bar{i}$-th equilibrium of any fixed sequence of equilibria satisfying equation (A.10) (this follows from Step 4, Lemma A.3 and the fact that, by equation (A.8), the payoff loss is increasing in γ_0).

Step 7: Proof of Proposition 3.2. Steps 4 and 6 establish that equation (A.2) holds in a sequence of efficient equilibria. This equation is equivalent to equation (3.7), and so Proposition 3.2 holds.

Step 8: Proof of Proposition 3.3. The proof follows immediately from Step 6 and equation (A.8). □

A.3 Proofs of the results in Section 4

Proof of Proposition 4.1

Proof. Solving equation (3.9) we have that the coarseness of communication established in Proposition 3.1 can be written as

$$c(\theta;b) = \frac{\int_{\theta^*}^{\theta} 8\, b(\theta')\, f(\theta')^{2/3}\, \mathrm{d}\theta'}{f(\theta)^{2/3}} \ ,$$

where we make the dependence on the bias function explicit.

Let $\hat{b}$ be a reduction of b in some set $\Theta' \subset \Theta$ which does not change the communicable state θ^*. Assume first that $\Theta' \subset [\theta^*, 1]$ and b is inward with respect to θ^* in Θ', that is, $b(\theta) \leq 0$ for all $\theta \in \Theta'$. Then, it is clear that $c(\theta;\hat{b}) = c(\theta;b)$ for all $\theta < \inf(\Theta')$, and $c(\theta;\hat{b}) \geq c(\theta;b)$ for all $\theta \geq \inf(\Theta')$. The result for a general Θ' and for an outward bias are proven analogously. □

Proof of Proposition 4.2

Proof. Assume b is increasing. Switching roles and reversing the state space if necessary, assume that also $b(1) > 0$.[23] This implies that $\theta^* < 1$, and that $\theta^* > 0$ only if $b(0) < 0$. Also notice that the bias is outward from the communicable state in this case, so using the same notation as in the proof of Proposition 4.1 we have $c(\theta;b) \leq \max\{c(0;b), c(1;b)\}$. Notice further that, using equation (3.9), since the distribution of states is uniform we have that $c(\cdot;b) + c(\cdot;-b)$ is constant, and then necessarily equal to $\max\{c(0;b), c(1;b)\}$. Finally, we have that $c'(\cdot;b)$ is increasing for $\theta > \theta^*$ and decreasing for $\theta < \theta^*$, so $c(\cdot;b)$ is convex. The result is then clear, since

$$\int_0^1 c(\theta;b)\, \mathrm{d}\theta \leq \int_0^{\theta^*} \frac{c(0;b)\, (\theta^* - \theta)}{\theta^*}\, \mathrm{d}\theta + \int_{\theta^*}^1 \frac{c(1;b)\, (\theta - \theta^*)}{\theta^*}\, \mathrm{d}\theta \leq \frac{\max\{c(0;b), c(1;b)\}}{2} \ .$$

Hence, since $c(\cdot;b) + c(\cdot;-b) = \max\{c(0;b), c(1;b)\}$, we have

$$\int_0^1 c(\theta;-b)\, \mathrm{d}\theta = \max\{c(0;b), c(1;b)\} - \int_0^1 c(\theta;b)\, \mathrm{d}\theta \geq \frac{\max\{c(0;b), c(1;b)\}}{2} \ .$$

The inequalities are strict if b is strictly increasing. □

[23] If $b(1) \leq 0$ (so $b(0) < 0$) then switching roles and reversing the state space (that is, applying the map $\theta \mapsto 1 - \theta$) generates the bias $\hat{b}(\theta) \equiv -b(1-\theta)$, which is increasing and satisfying $\hat{b}(1) > 0$.

B Extensions

This appendix extends our analysis in two directions. First, it generalizes the results in Section 3 to a more general set of payoff functions. Second, it discusses the limits of non-efficient equilibria as the bias gets small. The proofs of the results are in Section B.3.

B.1 General payoff functions

This section illustrates how the analysis of the small bias case can be extended to general payoff functions. Thus, we consider the same setting as in Section 2, the only difference being in the payoff functions of both the sender and the receiver.

Fix an outcome of the game $(\theta, m, a) \in \Theta \times M \times \mathbb{R}$. Now, the payoff of the sender is $u^s(\theta, a - \beta\, b(\theta))$, while the payoff of the receiver is $u^r(\theta, a)$, where $b : \Theta \to \mathbb{R}$ satisfies the same conditions as in our base model. We assume that, for each $\theta \in \Theta$, $u^s(\theta, \cdot)$ and $u^r(\theta, \cdot)$ are single peaked, and that both are maximized when the second component is θ, that is, $u^s(\theta, a - \beta\, b(\theta))$ is maximized when $a = \theta + \beta\, b(\theta)$, while $u^r(\theta, a)$ is maximized at $a = \theta$.[24] Also, without loss of generality, it is convenient to normalize $u^s(\theta,\theta) = u^r(\theta,\theta) = 0$ for all $\theta \in \Theta$. We assume that both u^s and u^r are three-times differentiable in each variable, and we define, for each $k \in \{0,1,2,3\}$,

$$u^s_k(\theta) \equiv \frac{1}{k!} \frac{\partial^k u^s(\theta, a)}{\partial a^k}\Big|_{a=\theta+\beta\, b(\theta)} \quad \text{and} \quad u^r_k(\theta) \equiv \frac{1}{k!} \frac{\partial^k u^r(\theta, a)}{\partial a^k}\Big|_{a=\theta},$$

and assume that $u^r_2(\theta), u^s_2(\theta) < 0$ for all $\theta \in \Theta$. Finally, to ensure that all equilibria are essentially equivalent to partition equilibria, we assume that $\frac{\mathrm{d}^2}{\mathrm{d}\theta \mathrm{d}a} u^s(\theta, a - \beta\, b(\theta)) > 0$ and $\frac{\mathrm{d}^2}{\mathrm{d}\theta \mathrm{d}a} u^r(\theta, a) > 0$.

The next result generalizes Proposition 3.1 by characterizing the coarseness of communication c for general payoff functions, and also generalizes the result in Proposition 3.3 for the equilibrium payoff.

Proposition B.1. *Assume that the payoff loss functions of the sender and the receiver satisfy the conditions above. Let $c \in \mathscr{C}^1(\Theta, \mathbb{R}_+)$ be the unique solution of*

$$c'(\theta) = 8\, b(\theta) - \frac{2\, c(\theta)}{3} \Big(\frac{f'(\theta)}{f(\theta)} + \underbrace{\frac{u^{r\prime}_2(\theta)}{u^r_2(\theta)}}_{(*)} + \underbrace{\frac{u^r_3(\theta)}{2u^r_2(\theta)} - \frac{u^s_3(\theta)}{2u^s_2(\theta)}}_{(**)} \Big) \tag{B.1}$$

satisfying that $c(\theta) \geq 0$ for all $\theta \in \Theta$ and $c(\theta^*) = 0$ for a (generically unique) state $\theta^* \in \Theta$. Then, for any efficient equilibrium, we have $\beta^{-1}\, (\overline{\tau}(\theta) - \underline{\tau}(\theta))^2 \approx c(\theta)$. Furthermore, the normalized equilibrium

[24]Our parametrization, which is innocuous for a fixed bias function b, is convenient to obtain results "for small biases" while keeping the rest of the primitives (and, in particular, the shape of the payoff functions around the ideal action) fixed.

payoffs are given by

$$U^s \approx \int_0^1 u_2^s(\theta)\, \frac{c(\theta)}{12}\, f(\theta)\, \mathrm{d}\theta \quad \textit{and} \quad U^r \approx \int_0^1 u_2^r(\theta)\, \frac{c(\theta)}{12}\, f(\theta)\, \mathrm{d}\theta\ . \tag{B.2}$$

There are two extra terms $(*)$ and $(**)$ inside the parenthesis on the right hand side of equation (B.1) with respect to the right hand side of equation (3.7). These terms incorporate, respectively, the effect of the change in the curvature of the payoff loss function of the receiver, and the effect the skewness of both payoff loss functions.

The effects on equilibrium communication of changes in the (state-dependent) curvature of the payoff function of the receiver, u_2^r, are similar to those of a state-dependent likelihood of the states. To see this, assume that the distribution of states is locally uniform, $f'(\theta) = 0$, and the convexity of the payoff loss function of the receiver (or her risk aversion) is increasing (in absolute value) in θ, so its derivative is negative, $u_2^{r\prime}(\theta) < 0$. This implies, similarly to the case where the likelihood of the states is increasing, that action taken by the receiver in equilibrium within the interval of the partition containing θ is relatively high. The reason is that when $u_2^{r\prime}(\theta) < 0$, the higher the realization of the state of the world, the higher the receiver's payoff loss from taking an action at a given distance from it. As a result, the receiver takes a high action in order to avoid a large loss in the case that the state of the world is in the high side of the interval.

The skewness terms of the sender and the receiver push the derivative of the coarseness of communication in opposite directions. Consider, for example, $u_3^r(\theta) < 0$ for some $\theta \in \Theta$. In this case the payoff loss of the receiver is skewed to the right: for a fixed action a, the payoff loss is higher if the state of the world is $a + \varepsilon$ than when it is $a - \varepsilon$, for $\varepsilon > 0$ small. This induces the receiver to take a relatively high action in the interval of the partition containing θ. Analogously, assume $u_3^s(\theta) < 0$, so when the realized state is $\theta \in \Theta$, the sender's payoff loss is higher if the action of the receiver is $\theta + \beta\ b(\theta) + \varepsilon$ than if it is $\theta + \beta\ b(\theta) - \varepsilon$. Similarly to when the bias is positive, the action of the receiver needs to be shifted to the left in order to keep the sender's indifference condition at the boundary of equilibrium intervals.[25]

We see in equation (B.2) that, for a fixed c, the payoff losses of the sender and the receiver only differ because of the different curvature of their payoff functions, u_2^t. Now, in addition to the coarseness with which a state θ is communicated and its likelihood, the contribution of a state to the payoff loss is multiplied by the local sensitivity of the payoff to deviations from the optimal

[25]Notice equation (B.1) does not contain any term similar to $(*)$ involving, instead, the curvature of the sender's payoff function $u_2^s(\theta)$. The reason is that the indifference condition of the sender at the boundary of two equilibrium intervals is not directly affected by the symmetric part of its payoff function. If, for example, the payoff function $u^s(\theta, \cdot)$ is symmetric around $\theta + \beta\ b(\theta)$ for any θ, the indifference condition only requires $a(\theta_{n-1}, \theta_n)$ and $a(\theta_n, \theta_{n+1})$ to be at the same distance of $\theta_n + \beta\ b(\theta_n)$, independently of the state-dependent curvature.

action. In this case, even though the sender and the receiver's payoff functions differ, they both benefit from a better communication: both payoffs are decreasing in the coarseness of communication c.

B.2 Inefficient equilibria

The main focus of our previous analysis is on characterizing efficient equilibria. As we argued, focussing on this class of equilibria is useful to obtain the limits of strategic communication and to compare equilibrium outcomes for different bias functions. Still, the CS model features a large number of equilibria, which can be grouped, for each fixed primitives of the model, in terms of "essential equivalence" (i.e., same joint distribution of states and actions) in a discrete set of "equivalence classes", each of them containing one partition equilibrium.

When the bias is small, efficient equilibria satisfy equation (3.3). The proofs of Propositions 3.1 and 3.2 additionally show that sequences of equilibria featuring the maximum number of messages satisfy equation (3.11), that is, they are asymptotically efficient.[26] We now consider properties of equilibria with a lower number of messages. We do this by considering sequences of decreasing biases and corresponding equilibria where the number of messages increases at a rate lower than that for efficient equilibria.[27] The following result characterizes the limit behavior of the corresponding equilibria with an increasingly large number of used messages.

Proposition B.2. *Fix a sequence $(\beta^i)_i$ strictly decreasing toward 0 and a sequence of corresponding equilibria. Let N^i be number of messages used the i-th equilibrium, assume it is finite for each i, and that $N^i \to \infty$ as $i \to \infty$. Then, $\lim_{i\to\infty}(\beta^i)^{1/2}\, N^i < \infty$, and*

1. *if $\lim_{i\to\infty}(\beta^i)^{1/2}\, N^i = k$ for some $k>0$, then $(\beta^i)^{-1}\,(\overline{\tau}^i(\theta)-\underline{\tau}^i(\theta))^2 \approx c_k(\theta)$ as $i\to\infty$, where c_k is the solution of (3.9) satisfying $\int_0^1 c_k(\theta)^{-1/2}\,\mathrm{d}\theta = k$, and*
2. *if $\lim_{i\to\infty}(\beta^i)^{1/2}\, N^i = 0$, then there exists some $k'>0$ such that*

$$\lim_{i\to\infty} \frac{(\overline{\tau}^i(\theta)-\underline{\tau}^i(\theta))\, N^i}{f(\theta)^{1/3}} = k' \text{ for all } \theta\in\Theta.$$

   In this case, the equilibrium payoff loss of both the sender and the receiver are $O(N^{-i}) > O((\beta^i)^{1/2})$ as $i\to\infty$.

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[26] In the CS model it is ensured that, under the so-called condition M, it is the case that both the sender and the receiver prefer equilibria with more interval partitions. In our model this may not hold but, as we show, equilibria with the most number of partitions give the seller and the buyer the same asymptotic payoff than efficient equilibria.

[27] The CS model with a fixed number of messages $N\in\mathbb{N}$ and no conflict of interest is analyzed in Sobel (2015). See also Dilmé (2018) for an analysis of costly information transmission without conflict of interest and a large number of messages available for communication.

Proposition B.2 characterizes equilibrium communication depending on how the number of messages relates to the bias size. It first considers the case where the number of messages used in equilibrium increases fast, that is, the square size of the partition elements remains $O(\beta^{i})$ (as in a sequence of efficient equilibria). In this case, the coarseness of communication still follows equation (3.9), but the coarseness of communication is higher than for efficient equilibria and, as a result, there may not be a communicable state. If, instead, the number of messages used in communication increases at a rate slower than the square root of the bias, the bias becomes increasingly less important to determine equilibrium information transmission. In this case the precision with which a state is communicated is approximated by a function of its likelihood. The equilibrium communication becomes increasingly close to optimal information transmission in the absence of conflict of interest, but with a limited number of messages.

B.3 Proofs of the results in Appendix B

Proof of Proposition B.1

Proof. The proof is analogous to the proof in Appendix A.2. The proof only changes due to the fact that there is an additional term in equation (3.7) owed to the fact that the receiver's payoff function curvature is state-dependent and it is skewed, so we have

$$\beta^{-1}\left(\alpha(\theta)-\frac{\overline{\tau}(\theta)+\underline{\tau}(\theta)}{2}\right)\approx\underbrace{\frac{1}{12}\left(\frac{f'(\theta)}{f(\theta)}+\frac{u_2^{r\prime}(\theta)}{u_2^r(\theta)}+\frac{u_3^r(\theta)}{2\,u_2^r(\theta)}\right)c(\theta)}_{\equiv\hat{\alpha}(\theta)},$$

and since the payoff function of the sender is skewed, the limit equation for indifference condition (3.8) now is given by

$$c'(\theta)=8\left(b(\theta)+\frac{1}{12}\,\frac{u_3^s(\theta)}{u_2^s(\theta)}\,c(\theta)-\hat{\alpha}(\theta)\right).$$

We compute the payoff of the receiver generalizing the argument in equation (A.8):

$$\begin{aligned}\int_0^1 u^r(\theta,\alpha(\underline{\tau}(\theta),\overline{\tau}(\theta)))\,f(\theta)\,\mathrm{d}\theta &= \int_0^1 u_2^r(\overline{\tau}(\theta))\left(\frac{\overline{\tau}(\theta)+\underline{\tau}(\theta)}{2}-\theta\right)^2 f(\theta)\,\mathrm{d}\theta+o(\beta)\\ &=\int_0^1 u_2^r(\overline{\tau}(\theta))\left(\frac{\overline{\tau}(\theta)+\underline{\tau}(\theta)}{2}-\theta\right)^2 f(\overline{\tau}(\theta))\,\mathrm{d}\theta+o(\beta)\\ &=\int_0^1 u_2^r(\overline{\tau}(\theta))\,\frac{\left(\overline{\tau}(\theta)-\underline{\tau}(\theta)\right)^2}{12} f(\overline{\tau}(\theta))\,\mathrm{d}\theta+o(\beta)\\ &=\frac{\beta}{12}\int_0^1 u_2^s(\theta)\,c(\theta)\,f(\theta)\,\mathrm{d}\theta+o(\beta)\end{aligned}$$

where, for the third equality, we used that $\int_{\underline{\tau}(\theta)}^{\overline{\tau}(\theta)}\left(\frac{\overline{\tau}(\theta)+\underline{\tau}(\theta)}{2}-\tilde{\theta}\right)^2\mathrm{d}\tilde{\theta}=\frac{1}{12}(\overline{\tau}(\theta)-\underline{\tau}(\theta))^2$. The expected payoff loss of the sender can be calculated analogously. □

Proof of Proposition B.2

Proof. See Appendix A.2. □

References

Admati, A. R., and P. Pfleiderer, 2004, "Broadcasting opinions with an overconfident sender," *International Economic Review*, 45(2), 467–498.

Alonso, R., and N. Matouschek, 2008, "Optimal delegation," *The Review of Economic Studies*, 75(1), 259–293.

Ambrus, A., and S. E. Lu, 2014, "Almost fully revealing cheap talk with imperfectly informed senders," *Games and Economic Behavior*, 88, 174–189.

Battaglini, M., 2004, "Policy advice with imperfectly informed experts," *Advances in theoretical Economics*, 4(1).

Blinder, A. S., M. Ehrmann, M. Fratzscher, J. De Haan, and D.-J. Jansen, 2008, "Central bank communication and monetary policy: A survey of theory and evidence," *Journal of Economic Literature*, 46(4), 910–945.

Blume, A., O. J. Board, and K. Kawamura, 2007, "Noisy talk," *Theoretical Economics*, 2(4), 395–440.

Born, B., M. Ehrmann, and M. Fratzscher, 2014, "Central bank communication on financial stability," *The Economic Journal*, 124(577), 701–734.

Chen, Y., 2011, "Perturbed communication games with honest senders and naive receivers," *Journal of Economic Theory*, 146(2), 401–424.

Chen, Y., and S. Gordon, 2015, "Information transmission in nested sender–receiver games," *Economic Theory*, 58(3), 543–569.

Crawford, V. P., and J. Sobel, 1982, "Strategic information transmission," *Econometrica*, 50, 1431–1451.

Deimen, I., and D. Szalay, 2014, "Smooth, strategic communication," CEPR Discussion Paper No. DP10190.

——— , 2019, "Delegated expertise, authority, and communication," *American Economic Review*, 109(4), 1349–74.

Dessein, W., 2002, "Authority and communication in organizations," *The Review of Economic Studies*, 69(4), 811–838.

Dilmé, F., 2018, "Optimal languages," University of Bonn, mimeo.

Dimitrakas, V., and Y. Sarafidis, 2005, "Advice from an expert with unknown motives," *Available at SSRN 797844*.

Eso, P., and Y.-f. Fong, 2008, "Wait and see," working paper.

Gordon, S., 2010, "On infinite cheap talk equilibria," University of Montreal, mimeo.

Iserles, A., 2009, *A first course in the numerical analysis of differential equations*. Cambridge University Press.

Kartik, N., 2009, "Strategic communication with lying costs," *The Review of Economic Studies*, 76(4), 1359–1395.

Kartik, N., M. Ottaviani, and F. Squintani, 2007, "Credulity, lies, and costly talk," *Journal of Economic Theory*, 134(1), 93–116.

Kawamura, K., 2015, "Confidence and competence in communication," *Theory and Decision*, 78(2), 233–259.

Knütter, R., B. Mohr, and H. Wagner, 2011, *The effects of central bank communication on financial stability: a systematization of the empirical evidence*. Fernuniv.

Li, M., and K. Madarász, 2008, "When mandatory disclosure hurts: Expert advice and conflicting interests," *Journal of Economic Theory*, 139(1), 47–74.

Melumad, N. D., and T. Shibano, 1991, "Communication in settings with no transfers," *The RAND Journal of Economics*, pp. 173–198.

Morgan, J., and P. C. Stocken, 2003, "An analysis of stock recommendations," *RAND Journal of Economics*, pp. 183–203.

Moscarini, G., 2007, "Competence implies credibility," *The American Economic Review*, pp. 37–63.

Ottaviani, M., 2000, "The economics of advice," University College London, mimeo.

Ottaviani, M., and P. N. Sørensen, 2006, "The strategy of professional forecasting," *Journal of Financial Economics*, 81(2), 441–466.

Ottaviani, M., and F. Squintani, 2006, "Naive audience and communication bias," *International Journal of Game Theory*, 35(1), 129–150.

Sobel, J., 2015, "Broad Terms and Organizational Codes," UC San Diego, mimeo.

Spector, D., 2000, "Pure communication between agents with close preferences," *Economics Letters*, 66(2), 171–178.

Stein, J. C., 1989, "Cheap talk and the Fed: A theory of imprecise policy announcements," *The American Economic Review*, pp. 32–42.

Szalay, D., 2012, "Strategic information transmission and stochastic orders," working paper.