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The Role of Discounting in Bargaining with One-Sided Offers

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The Role of Discounting in Bargaining with One-Sided Offers

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This paper analyzes a continuous-time Coase setting with finite horizon, interdependent values, and different discount rates. Our full characterization of equilibrium behavior permits studying how patience shapes the bargaining outcome. We obtain that (i) the seller's commitment problem persists even when she is fully patient, (ii) making the seller more impatient may increase equilibrium prices, (iii) when adverse selection is not strong, the buyer is ex-post better off when he is more impatient, and (iv) when discounting is time-dependent, episodes where the seller or the buyer have a high discount rate feature a large probability of trade, but only periods with high buyer discounting lead to a fast price decline.

Keywords: Bargaining, one-sided offers, different discount factors.

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1 Introduction

Bargaining theory is central in economics. It helps understanding, among others, price formation in decentralized markets (Osborne and Rubinstein, 1990), negotiations between unions and firms (Hart, 1989, and Cramton and Tracy, 1992), cartel stability (Hnyilicza and Pindyck, 1976), or pretrial settlements (Spier, 1992). In studying bargaining, patience plays a critical role. Intuitively, the more patient an agent is, the lower is her delay cost, and hence the more credibility she has in sustaining a tough bargaining position. Rubinstein (1982) showed that, in a bargaining model with alternating offers and without private information, the payoff an agent obtains is larger when she is more patient, smaller when the other agent is more patient, and approaches the commitment payoff as she becomes fully patient.

This paper analyzes the role of patience in the other canonical bargaining setting: one-sided offers with asymmetric information. We study a version of Sobel and Takahashi (1983), Fudenberg and Tirole (1983), and Cramton (1984). There are a seller of an indivisible good and a buyer. The buyer's type is his private valuation for the good, v, which is drawn from an absolutely-continuous distribution on $[0, \overline{v}_0]$. The seller makes price offers to the buyer, who either accepts them or rejects them. We allow the discount (or interest) rates of the seller and the buyer —denoted r_{s} and r_{b}, respectively— to be different.

The main departures of our setting with respect to that in Sobel and Takahashi (1983), Fudenberg and Tirole (1983), and Cramton (1984) are the following. First, we study the case where the horizon T is arbitrary. This permits studying the commitment problem of the seller independently of her discount rate and broadens the set of applications where our results apply.[1] Second, we allow the seller's cost $c(v) \in [0, v]$ to be independent the buyer's valuation (private values case, as in Gul, Sonnenschein, and Wilson, 1986, where $c \equiv 0$), or to depend on the buyer's valuation (interdependent values case, as in Deneckere and Liang, 2006). Finally, we set the game in continuous time. This gives the model enough tractability to study the effect of discounting in the bargaining outcome.[2]

This paper's first contribution is providing a full characterization of the equilibrium dynamics in a bargaining model with one-sided and asymmetric model. We completely characterize the unique Markov perfect equilibrium with reservation prices in closed form, with the time t and the highest remaining valuation of the buyer $\overline{v}_t$ as state variables. We show that prices decrease smoothly over time. The seller screens the buyer slowly, there are no trade impasses, and the only trade burst occurs at the deadline.

[1]Bargaining with a deadline is frequent in practice. Fuchs and Skrzypacz (2013a) provide references to several empirical studies documenting "deadline effects" (last minute deals) in labor contract negotiations and civil cases.

[2]Even though there are recent bargaining models in continuous time (see the literature review below), this paper's model is, to our knowledge, the first to analyze the standard one-sided offers settings (Gul, Sonnenschein, and Wilson, 1986; and Deneckere and Liang, 2006) directly in continuous time.

The second contribution is documenting, making use of the tractability of our approach, how patience shapes the bargaining outcome. We obtain that the commitment problem of the seller is severe: independently of her and the buyer's impatience levels, the seller's payoff is equal to the one she obtains by just waiting until the deadline to sell at the monopolistic price. The seller's commitment problem does not vanish in the limit where r_{s} approaches 0 while keeping r_{b} fixed as her equilibrium payoff converges to the static monopolist payoff; while a seller with commitment obtains a higher payoff by slowly lowering the price over time (see Fudenberg and Tirole, 1983).

We derive the following simple and easily-interpretable expression for the equilibrium price offered in each state $(t, \overline{v}_t)$:

$$p(t, \overline{v}_t) - c(\overline{v}_t) = e^{-r_{\mathrm{s}}(T-t)} \left(p^*(\overline{v}_t) - c(\overline{v}_t) \right) , \tag{1}$$

where $p^*(\overline{v}_t)$ is the static monopolistic price when the buyer's valuation is known to be lower than $\overline{v}_t$. The seller's payoff from trading with the buyer with valuation $\overline{v}_t$ at time t is equal to the payoff she obtains from this buyer if she waits until time T and trades at the corresponding monopolistic price. In particular, the price at each state is independent of the buyer's discount rate. Also, as long as $p^*(\overline{v}) \geq c(\overline{v})$ for all $\overline{v}$ —we say that *adverse selection is not strong* in this case—, an increase in either r_{s} or T lowers the price at each state (since the right-hand side of equation (1) is positive), and this is shown to increase the buyer's payoff independently of his valuation for the good. Instead, if for example $p^*(\overline{v}_0) < c(\overline{v}_0)$, the seller makes initial losses by setting prices below the cost, which are compensated by later sales to lower-cost buyer types. An increase in the seller's interest rate makes her less willing to do so, so she slows down initial sales by charging higher a price early in the game. This makes the seller and some types of the buyer worse off. Similarly, enlarging the time horizon worsens the seller's commitment problem, hence harming the profitability of later sales and, under strong adverse selection, increasing initial prices and making the seller and some buyer types worse off.

We characterize the equilibrium price dynamics, which are driven by the buyer's incentive compatibility condition $\dot{p}_t = -r_{\mathrm{b}} (\overline{v}_t - p_t)$. The equilibrium price rapidly decreases over time when r_{b} is large, while it is approximately constant when r_{b} is small. Intuitively, the refusal of accepting a given price offer is a stronger signal of lower valuation when the buyer is more impatient, and this induces the seller to decrease the price faster. When adverse selection is not strong, there is an additional effect: since each threshold valuation $\overline{v}$ is reached sooner, the price each buyer accepts in equilibrium is lower, hence increasing the difference between $\overline{v}_t$ and p_t at the acceptance time. The additional increase in the speed of price decline implies that the buyer is better off when he is more impatient *independently of his valuation*: the reduction in the delay and price more than compensates for the higher cost of waiting. This result shows that an agent may benefit from having a more pressing need for an early agreement (e.g., facing a higher interest rate). In other words, using impatience as a measure of bargaining power may not be appropriate in bargaining settings with asymmetric information.

We shed some light on our results by comparing them with the case where the seller has full commitment power. We do so in an example that can be solved in close form. We observe that when the seller is more patient than the buyer, giving the seller's commitment power reduces the probability of trade but, unlike the case where the seller and the buyer are equally patient, it increases delay.

Finally, motivated by the analysis in Hart (1989), we extend the model to the case where discount rates are time-dependent. We obtain that, in the private-values case, episodes of either large seller discounting or large buyer discounting feature faster screening, but only episodes of large buyer discounting imply faster price decrease. If, for example, both discount rates increase over time, trade speeds up and price decline is faster towards the deadline, indicating that periods with a large delay cost feature a large probability of agreement.

Literature review: Our paper contributes to the literature on bargaining with asymmetric information, reviewed in Ausubel, Cramton, and Deneckere (2002) and Fuchs and Skrzypacz (2020). To our knowledge, the analysis of the role of discounting in bargaining has only been addressed by Sobel and Takahashi (1983) and Evans (1989). Sobel and Takahashi study two-period and infinite-horizon versions of a bargaining model with private values. Consistently with our findings, they obtain that the buyer (seller) benefits from an increase in the seller's (buyer's) impatience in both versions. Evans studies a two-type model with independent values. He shows that, if the buyer is more patient than the seller, there is a trade impasse. Our model does not feature such impasse because we study the so-called no-gap case and the horizon is finite.

Other papers have studied bargaining with a deadline. Most saliently, Fuchs and Skrzypacz (2013a) study how the bargaining outcome's efficiency depends on the deadline and the disagreement payoff when agents are equally patient. They obtain that a smaller disagreement payoff induces more trade before the deadline, while the length of the deadline may affect efficiency non-monotonically.[3]

We also contribute to the recent literature modeling bargaining directly continuous time. Examples include Ortner (2017) (time-varying seller's private costs), Daley and Green (2020) and Lomys (2020) (bargaining with learning), and Chaves (2020) (bargaining with arrival of new traders). We provide a new approach to defining strategies and the corresponding outcome. In particular, our strategies do not restrict the set of possible outcomes (do not require right-continuity or monotonicity of prices, for example). Furthermore, our Markov strategies depend on time, given that our horizon is finite.

The rest of the paper is organized as follows: Section 2 presents our continuous-time model, Section 3 contains the equilibrium analysis, Section 4 provides the comparative-statics results regarding the role of discounting in bargaining, and Section 5 concludes. The Appendix contains the proofs of the results.

[3]Some other bargaining models with finite horizon are Ma and Manove (1993), Fershtman and Seidmann (1993), Thépot (1998), and Fanning (2016). Also, see Ausubel and Deneckere (1989) and Fuchs and Skrzypacz (2013b) for papers analyzing the so-called "no-gap" case in the private and interdependent values cases, respectively.

2 Model

We study a continuous-time bargaining model with finite horizon, where time belongs to $[0, T]$. There is a seller of a durable good and a buyer. The buyer's private valuation for the good, v, is distributed according to some distribution F with a continuous and positive probability density function f and with support equal to $[0, \overline{v}_0] \subset \mathbb{R}_+$, for some $\overline{v}_0 > 0$. The seller's valuation for the good is $c(v)$ where $c: [0, \overline{v}_0] \to \mathbb{R}$ is a continuously differentiable and non-decreasing function satisfying that $c(v) \in [0, v)$ for all $v \in (0, \overline{v}_0]$.[4] Hence, there is common knowledge of gains from trade, and there is no gap for lower buyer valuations. At each instant $t \in [0, T]$, the seller makes an offer, and the buyer decides to accept it or not. The game ends when the buyer accepts an offer or when the deadline is reached. Both the seller and the buyer are risk neutral, and their discount (or interest) rates are $r_{\mathrm{s}} > 0$ and $r_{\mathrm{b}} > 0$, respectively.

We make two assumptions throughout the paper:

Assumption 1. For any $\overline{v} \in (0, \overline{v}_0]$, the function $p \mapsto \int_p^{\overline{v}} (p - c(v))\, F(\mathrm{d}v)$ has a unique maximizer, which will henceforth be denoted by $p^*(\overline{v})$.

Assumption 2. For any $\overline{v} \in [0, \overline{v}_0]$, $c(\overline{v}) \leq \frac{r_{\mathrm{b}}}{r_{\mathrm{s}}} \overline{v}$.

The first assumption is standard. The second assumption is discussed and relaxed in Section 3.3. Note that Assumption 2 is always satisfied in the private values case (i.e., when $c(\overline{v}) = 0$ for all $\overline{v} \in (0, \overline{v}_0]$), and also when the seller is more patient than the buyer (i.e., when $r_{\mathrm{s}} \leq r_{\mathrm{b}}$).

We now proceed to formally define the continuous-time game:

Histories: A *history* is a measurable function from $[0, t]$ to $\mathbb{R}$, for some $t \in \{0^-\} \cup [0, T]$, generically denoted $p^t \in \mathbb{R}^{[0,t]}$, where p^{0^-} is the empty history and $[0, 0^-] \equiv \emptyset$.

Seller's strategies: A *(pure) strategy for the seller* consists is a function P assigning, to each history p^t, a continuation price path $P(p^t) \in \mathbb{R}^{(t,T]}$ (where $(0^-, T] \equiv [0, T]$) such that

$$P_{t''}(p^t) = P_{t''}(p^t, P_{(t,t']}(p^t)) \ \text{ for all } t' > t \text{ and } t'' > t' \ , \tag{2}$$

where $P^{t''}(p^t)$ is the value $P(p^t)$ assigns to t''. Intuitively, the consistency condition (2) requires that the seller does not deviate from her continuation strategy. More formally, note that when the seller follows

[4] The assumption that the informed agent has superior knowledge about the uninformed agent's valuation is plausible in a number of applications of the model. For example, the durable good may be the procurement of a (legal/repairing/medical) service, so the buyer knows more about his idiosyncratic problem. Also, our setting is equivalent to one where the seller sells to a unit mass of buyers, and the cost of producing $1 - F(\overline{v})$ units is $\int_{\overline{v}}^{\overline{v}_0} c(v)\, F(\mathrm{d}v)$. In this case, the declining marginal cost can be associated to learning-by-doing, for example. Finally, analogous results can be obtained if the roles of the seller and the buyer are reversed; then, the positively correlated valuation may come from the underlying quality of the good known by the seller.

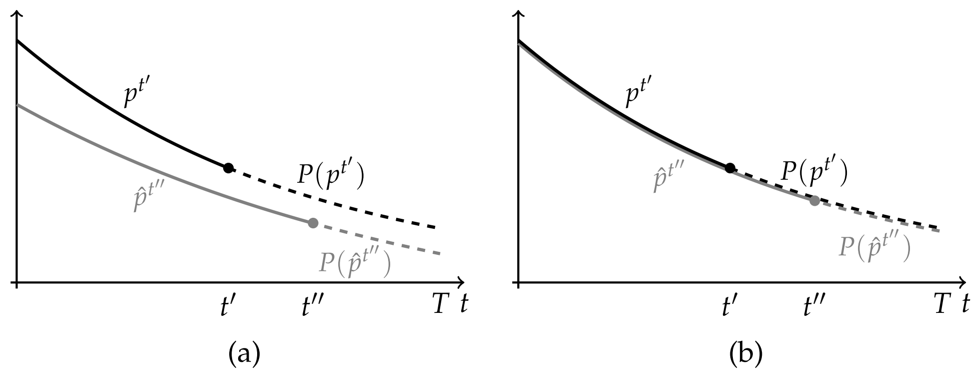

Figure 1: Both (a) and (b) provide examples of two price histories, $p^{t'}$ and $\hat{p}^{t''}$, and their corresponding continuation price paths, $P(p^{t'})$ and $P(\hat{p}^{t''})$. Note that, in (b), $\hat{p}^{t''}$ coincides with $P^{t''}(p^{t'})$, and hence the consistency condition (2) imposes that $P^T(p^{t'}) = P^T(\hat{p}^{t''})$.

the strategy on $(t, t']$ after history p^t, the history at time t' is $(p^t, P^{t'}(p^t))$. Then, the condition requires that the planned price at time $t'' > t'$ is the same under the continuation strategies at p^t and $(p^t, P^{t'}(p^t))$ (see Figure 1 for an illustration). Abusing notation, for each $t' \leq t$, $P_{t'}(p^t)$ is the price specified by p^t at time t'. Similarly, $P^{t'}(p^t)$ denotes $P_{[0,t']}(p^t) \in \mathbb{R}^{[0,t']}$ for any $t' \in \{0^-\} \cup [0, T]$. Condition (2) can then be written as "$P^T(p^t) = P^T(P^{t'}(p^t))$ for all $t' > t$."

Our definition of the seller's strategy is a modeling innovation of this paper and it is convenient for the following reasons. First, it does not impose any monotonicity or regularity condition on the choice of the price path. The price outcome from strategy P (until the buyer accepts) is $P(\emptyset)$, which can be any measurable function from $[0, T]$ to $\mathbb{R}$. The same applies for the continuation play after any history. Second, it is suitable writing strategies recursively, as it specifies a unique continuation strategy after each on- or off-path history p^t, which is fully described by $P(p^t)$.[5]

Buyer's strategies: A *(pure) strategy for the buyer* specifies, for each history p^t and valuation v, an acceptance decision $a^v(p^t) \in \{0, 1\}$, where $a^v(p^t) = 1$ means "accept" and $a^v(p^t) = 0$ means "reject". We assume that $\{v | a^v(p^t) = 1\}$ is a measurable set for all p^t.

Outcome: Fix some strategy profile (P, a), a history p^t and a buyer's valuation v. Let

$$A^v(p^t; P, a) \equiv \{\, t' > t \,|\, a^v(P^{t'}(p^t)) = 1 \,\}$$

be the set of acceptance times after time t by the v-buyer. The *transaction time* that (P, a) generates after

[5]It is known that, when strategies are defined as a map from the previous history to the current action, they may generate non-unique outcomes even when there is only one player taking actions. For example, if one specifies strategies as a map from previous prices to the current price, there are multiple outcomes consistent with the specification "$P_t = 0$ if the price is 0 at all times in $[0, t)$ and $P_t = 1$ otherwise".

p^t for the buyer with valuation v is equal to

$$t^v(p^t; P, a) \equiv \inf(A^v(p^t; P, a)) \in [t, T] \cup \{+\infty\} ,$$

where $t^v(p^t; P, a) = +\infty$ means that the buyer with valuation v rejects all offers after time t. If $t^v(p^t; P, a) < +\infty$, the *transaction price* for the v-buyer that (P, a) generates after p^t is equal to

$$p^v(p^t; P, a) \equiv \begin{cases} P_{t^v(p^t;P,a)}(p^t) & \text{if } t^v(p^t; P, a) \in A^v(p^t; P, a), \\ \lim_{t' \searrow t^v(p^t;P,a)} \inf \left(P_{(t^v(p^t;P,a),t']\cap A^v(p^t;P,a)}\right) & \text{otherwise.} \end{cases}$$

The inferior limit guarantees that the transaction price is uniquely defined. The choice of the inferior limit (instead of the superior limit, or a combination of them) is innocuous for the equilibrium analysis. Whenever the strategy profile and history are clear, we will not explicitly write the dependence of t^v and p^v on it.

Payoffs: Fix a strategy profile (P, a), a history p^t, and a buyer's valuation v. The *realized continuation payoff of the seller* is $e^{-r_s(t^v - t)}(p^v - c(v))$ and the *continuation payoff of the buyer* is $e^{-r_b(t^v - t)}(v - p^v)$.

2.1 Equilibrium concept

Perfect Bayesian equilibrium

We now define perfect Bayesian equilibria in the usual way. A *belief process* is a function $F(\cdot|\cdot)$ assigning, to each history p^t a posterior belief $F(\cdot|p^t) \in \Delta([0, \overline{v}_0])$.

Definition 2.1. A *(pure-strategy) perfect Bayesian equilibrium (PBE)* is a strategy profile (P, a) and a belief process $F(\cdot|\cdot)$ such that:

1. The seller's strategy P maximizes the seller's expected continuation payoff after each history p^t, given the buyer's strategy and the belief $F(\cdot|p^t)$; that is, maximizes

$$\int_0^{\overline{v}_0} e^{-r_s(t^v(p^t;P^\dagger,a)-t)} \left(p^v(p^t; P^\dagger, a) - c(v)\right) F(\mathrm{d}v|p^t)$$

   with respect to any seller's strategy $P^\dagger$.

2. For any v, the buyer maximizes his payoff after each history p^t; that is; maximizes

$$e^{-r_b(t^v(p^t;P,a^\dagger)-t)}(v - p^v(p^t; P, a^\dagger))$$

   with respect to any buyer's strategy $a^\dagger$.

3. Bayes' rule: $F(\cdot|\emptyset) = F(\cdot)$ and, for all histories p^t and $t' < t$, we have

$$F(v|p^t) = \frac{\int_0^v \mathbb{I}_{\{v''|a^{v''}(P^{t''}(p^t))=0\ \forall t''\in(t',t]\}}(v')\, F(\mathrm{d}v'|P^{t'}(p^t))}{\int_0^{\overline{v}_0} \mathbb{I}_{\{v''|a^{v''}(P^{t''}(p^t))=0\ \forall t''\in(t',t]\}}(v')\, F(\mathrm{d}v'|P^{t'}(p^t))}$$

whenever the denominator is not 0.[6]

It is convenient to state now a standard property of the buyer's equilibrium behavior in bargaining models, which will ease the definition of Markov perfect equilibria:

Lemma 2.1 (skimming property)**.** *In any PBE, the higher is the buyer's valuations, the earlier he earlier and the higher price is the price he pays; that is, t^v is decreasing in v and p^v is increasing in v for all p^t.*

As usual, the skimming property permits focussing on a simple class of belief processes. Namely, from now on, we focus on belief processes where (on- or off- the path) which are upper truncations of F. Also, without loss of generality, we focus on equilibria where, if trade is supposed to occur with probability one at some history and yet the buyer rejects, the seller believes that the buyer's valuation is 0.

We now define the upper bound on the support of the belief distribution after some history. Fixing a PBE and a history p^t, the upper bound on the distribution is obtained as follows:

$$\overline{v}(p^t) \equiv \sup(\mathrm{supp}(F(\cdot|p^t)))\ .$$

When the history p^t is clear, we will use $\overline{v}_{t'}$ to denote $\overline{v}(P^{t'}(p^t))$; that is, $\overline{v}_{t'}$ is the supremum of the support of the seller's belief at time t'. Note that, by the skimming property, $\overline{v}_{t'}$ is decreasing in t'. Finally note that optimality requires that $\overline{v}_T = p^*(\overline{v}_{T^-})$.

Markov perfect equilibrium

Definition 2.2. A *reservation-price Markov perfect equilibrium* is a PBE $(P, a, F(\cdot|\cdot))$ satisfying that

1. for all p^t and $\hat{p}^t$ such that $\overline{v}(p^t) = \overline{v}(\hat{p}^t)$, we have that $P_{t'}(p^t) = P_{t'}(\hat{p}^t)$ for all $t' > t$, and
2. for all p^t and $\hat{p}^t$, we have $t^v(p^t; \hat{P}, a) = t^v(\hat{p}^t; \hat{P}, a)$ for all $\hat{P}$ and $v \leq \min\{\overline{v}(p^t), \overline{v}(\hat{p}^t)\}$.

The first property is standard: the price (and seller's continuation strategy) at time t depends on the time and the highest remaining valuation. The second requirement is analogous to the usual requirement that the buyer uses a *reservation price strategy*. Indeed, buyer's decision on whether to trade or not

[6]As usual, for a set $A \subset \mathbb{R}$, $\mathbb{I}_A(\cdot)$ is the indicator function, where $\mathbb{I}_A(x) = 1$ if $x \in A$ and $\mathbb{I}_A(x) = 0$ otherwise. Hence, $\mathbb{I}_{\{v''|a^{v''}(P^{t''}(p^t))=0\ \forall t''\in(t',t]\}}(v') = 1$ if and only if the v'-buyer does not trade in $[t', t]$.

depends only on the time and the price offered at that time. From now on, we refer to reservation-price Markov perfect equilibria as just equilibria.

Our definition of equilibrium has a caveat: in principle, for a fixed equilibrium, there may be states $(t, \overline{v})$ which are never reached (not even after a seller's deviation). In such states, there is no optimality requirement in the continuation play. As a result, formal arguments need to be tailored only for states which are reachable (given the equilibrium), and this unnecessarily clutters the exposition. To keep the argumentation simple, we make the following change to our setting. From now on, we assume that there is some large $M > 0$ such that, if at time $t = 0$ (and only at this time) the seller sets a price that can be written as $-M + \overline{v}$ for some $\overline{v} \in [0, \overline{v}_0]$, then the buyer is "forced" to accept such price if his valuation is higher than $\overline{v}$, while otherwise he is forced to reject this price. By offering such price, the seller receives an additional lump-sum payoff of $-M$. Of course, this is irrelevant for equilibrium behavior: setting a negative price at time 0 is strictly dominated. Introducing this assumption, nevertheless, has the following effect: now, for any equilibrium, $t \in (0, T]$, and $\overline{v} \in [0, \overline{v}_0]$, there is a history p^t such that $\overline{v}(p^t) = \overline{v}$, and the strategy is sequential optimality afterwards.[7]

3 Analysis

3.1 Basic properties

We now state some important properties of equilibria. All statements apply to any equilibrium of the game (we omit writing "*In any equilibrium,*").

No silent period

We begin with an important property, sometimes called "no silent period" (especially in discrete-time models) or "no trade gaps".[8] It establishes that there are no intervals of time where the probability of trade is 0.

Proposition 3.1 (No silent period)**.** *There is no time interval with no trade; that is, there is no history p^{t_1} and $t_2 > t_1$ such that $\overline{v}_{t_1}(p^{t_1}) = \overline{v}_{t_2}(p^{t_1}) > 0$.*

Differently from some stationary settings, Proposition 3.1 is not immediate in our setting for two reasons. The first reason is that the horizon is finite: even though not trading for a time interval delays

[7] An example of a history p^t such that $\overline{v}(p^t) = \overline{v}$ is the following: at time 0, the price is $-M + \overline{v}$, while at any other time $t' \in (0, t)$, the price offer is unacceptable (above $\overline{v}_0$).

[8] Similar properties are found in models with arrival of buyers (Fuchs and Skrzypacz, 2010) and news arrival (Daley and Green, 2020).

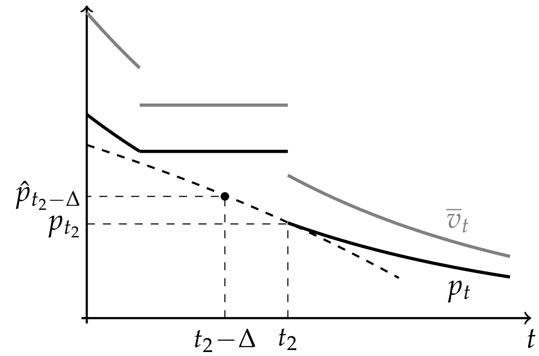

Figure 2: Illustration of the argument ruling out time intervals without trade. The dashed curve indicates the indifference curve for the $\overline{v}_{t_2}$-buyer. Hence, the seller can sell to the $\overline{v}_{t_2}$-buyer earlier (at time $t_2-\Delta$ instead of t_2) na at a higher price ($\hat{p}_{t_2-\Delta}$ instead of p_{t_2}), but also has to pay the cost $c(\overline{v}_{t_2})$ earlier.

revenue, it also shortens the time left until the deadline, and so it reduces the commitment problem of the seller. We show that the fact that the price declines on the path of play is enough to guarantee that the seller expected revenue increases if she sells earlier.

The second reason is that, in the interdependent-values case, trading earlier increases the present value of the revenue, but also the present value of the cost of selling. If the seller sells earlier to the buyer with valuation v at a price which is lower than the cost $c(v)$, selling earlier may be detrimental. This may occur, for example, when the buyer is patient, since then the price the seller can set to induce the buyer to buy earlier cannot be much higher than the equilibrium purchasing time.

We now provide an intuition for Proposition 3.1, depicted in Figure 2. To do so, fix an equilibrium. We argue using contradiction: we assume that the is an interval of time without trade (t_1, t_2), and then show that the seller has the incentive to speed trade in this interval. Let p_{t_2} be the equilibrium price at time t_2. Fix some small $\Delta > 0$, and let $\hat{p}_{t_2-\Delta}$ be the price that makes the $\overline{v}_{t_2}$-buyer indifferent between buying at time $t_2-\Delta$ and buying at time t_2. We compare the seller's discounted payoffs from selling to the $\overline{v}_{t_2}$-buyer at t_2 at price p_{t_2} and from selling to the $\overline{v}_{t_2}$-buyer at $t_2-\Delta$ at price $\hat{p}_{t_2-\Delta}$; that is,

$$e^{-r_s\Delta}\left(p_{t_2} - c(\overline{v}_{t_2})\right) \quad \text{vs} \quad \underbrace{(1-e^{-r_b\Delta})\,\overline{v}_{t_2} + e^{-r_b\Delta}\,p_{t_2}}_{=\hat{p}_{t_2-\Delta}} - c(\overline{v}_{t_2})\ . \tag{3}$$

The gain from early sale is lowest when p_{t_2} is equal to 0. In this case, the benefit from selling earlier is $(1-e^{-r_b\Delta})\,\overline{v}_{t_2}$ while the additional cost is $(1-e^{-r_s\Delta})\,c(\overline{v}_{t_2})$. Taking the limit $\Delta \to 0$, earlier purchase is beneficial if $c(\overline{v}_{t_2}) \leq \frac{r_b}{r_s}\,\overline{v}_{t_2}$; that is, if Assumption 2 is satisfied. This is shown to imply that, if Assumption 2 holds, Proposition 3.1 holds as well. See Section 3.3 for a further discussion.

Proposition 3.1 can be used to determine a transaction price for each state $(t, \overline{v})$. Indeed, fix a history p^t such that $\overline{v}_t(p^t) = \overline{v}$. Since there is no trade gap, the buyer with the highest valuation trades immediately; that is, $t^{\overline{v}} = t$. We can then define, for each pair $(t, \overline{v})$ and $t' > t$,

$$p(t, \overline{v}) \equiv p^{\overline{v}}(p^t) \text{ and } p_{t'}(t, \overline{v}) \equiv p^{\overline{v}}(P^{t'}(p^t)) \ .$$

As usual, when $(t, \overline{v})$ is clear, we will use $p_{t'} = p_{t'}(t, \overline{v}) = p(t', \overline{v}_{t'})$ to denote the price at time $t' > t$ under the assumption that the seller did not deviate after time t. Note that if follows immediately from Proposition 3.1 and the optimality of the buyers' strategy that $p_{t'}$ is continuous and decreasing in t' on $[t, T]$. We use $\Pi(t, \overline{v})$ to denote the continuation payoff of the seller in state $(t, \overline{v})$, and

$$\pi(t, \overline{v}) \equiv p(t, \overline{v}) - c(\overline{v})$$

to denote the surplus the seller obtains from the buyer with valuation $\overline{v}$ in state $(t, \overline{v})$.

Seller's equilibrium payoff

We proceed by presenting two results characterizing the equilibrium payoff of the seller. They can both be interpreted in light of the seller's willingness to speed or slow screening.

Proposition 3.2. *For any $(t, \overline{v})$ with $t < T$, the seller's payoff satisfies*

$$\Pi(t, \overline{v}) = \frac{1}{F(\overline{v})} \int_0^{\overline{v}} (p(t, \overline{v}) - c(\overline{v}))\, F(\mathrm{d}v) \ . \tag{4}$$

An interpretation Proposition 3.2 is the following. Assume the seller deviates and decreases the price very fast (but continuously) after time t. For example, as illustrated in Figure 3, she could set a price $\hat{p}_{t'} = \frac{t+\varepsilon-t'}{\varepsilon}\, p(t, \overline{v})$ for all $t' \in (t, t+\varepsilon]$ for some small $\varepsilon > 0$, so trade would occur for sure before time $t+\varepsilon$. Then, under this deviation, each buyer type $v \in [0, \overline{v}]$ would buy at a price approximately equal to $p(t, \overline{v})$ (the continuity of $p(\cdot, \overline{v})$ is shown in the proof of Proposition 3.2). The seller's payoff from such a deviation would then be approximately equal to the right-hand side of expression (4). Hence, Proposition 3.2 can be interpreted as establishing that, in equilibrium, the seller is willing to screen the buyer "infinitely fast."

While the previous result establishes that seller is willing to screen the buyer fast, the following result states that the seller is also willing to not screen the buyer at all. More formally, the seller's equilibrium payoff in state $(t, \overline{v})$ coincides with the payoff she would obtain if she would make unacceptable offers (above $\overline{v}$, for example) until the deadline and then, at the deadline, she would charge the monopolistic price $p^*(\overline{v})$.[9]

[9] Fuchs and Skrzypacz (2013a) obtain an analogous result in a model with private values, where the buyer's distribution follows a power distribution ($F(v) = v^a$ for $v \in [0, 1]$), and where equal impatience levels ($r_\mathrm{s} = r_\mathrm{b}$).

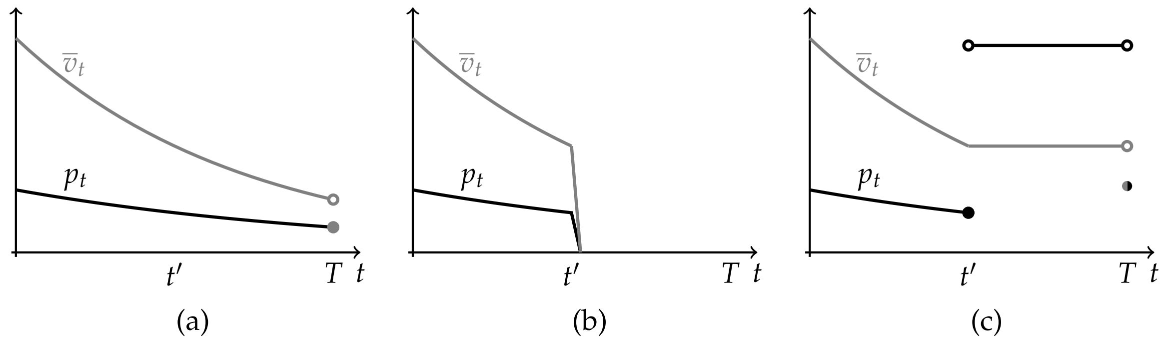

Figure 3: (a) depicts an example of an equilibrium path for p and $\overline{v}$. In (b), the seller deviates at time t' by lowering the price very fast after ward, and hence the buyer is screened very fast. In (c), the seller charges unacceptable prices in (t', T) and charges $p^*(\overline{v}_T)$ at time T. Propositions 3.2 and 3.3 establish that all three strategies give the same payoff to the seller.

Proposition 3.3. *For any $(t, \overline{v})$ with $t < T$, the seller's payoff equals the payoff she obtains from charging an unacceptable price until time T and then charging $p^*(\overline{v})$, i.e.,*

$$\Pi(t, \overline{v}) = e^{-(T-t)\, r_s}\, \Pi^*(\overline{v}) \ , \tag{5}$$

where $\Pi^*(\overline{v}) \equiv \int_{p^*(\overline{v})}^{\overline{v}} (p^*(\overline{v}) - c(v))\, F(\mathrm{d}v)$ equals the "static" monopolistic payoff.

We now provide a heuristic intuition for Propositions 3.2 and 3.3 (see Figure 3). We do so by using the standard Bellman equation, without proving that it holds:[10]

$$r_s\, \Pi(t, \overline{v}) = \tfrac{\partial}{\partial t} \Pi(t, \overline{v}) + \Big(f(\overline{v})\, \pi(t, \overline{v}) + \tfrac{\partial}{\partial \overline{v}} \Pi(t, \overline{v}) \Big)\, \dot{\overline{v}}(t, \overline{v}) \tag{6}$$

where, abusing notation, $\dot{\overline{v}}(t, \overline{v})$ denotes the speed at which the marginal valuation changes at state $(t, \overline{v})$. We can heuristically think of the seller's problem as one where she chooses, at each instant, the value of $\dot{\overline{v}}(t, \overline{v})$ (by deciding how fast the price falls). Using Proposition 3.1, we obtain that choosing $\dot{\overline{v}}(t, \overline{v}) = 0$ (i.e., not screening at all) cannot be strictly optimal. Similarly, if setting $\dot{\overline{v}}(t, \overline{v}) = +\infty$ would be strictly optimal for all $\overline{v}$, then the price would have to be equal to 0 (since screening would be very fast, and the buyer with valuation v never buys at a price above v). As a result, and given the linearity of the right-hand side of equation (6) in $\dot{\overline{v}}(t, \overline{v})$, it must be that $f(\overline{v})\, \pi(t, \overline{v}) + \frac{\partial}{\partial \overline{v}} \Pi(t, \overline{v}) = 0$; which implies equation (4). Hence, the seller is indifferent between screening very fast or very slow (or at any intermediate rate).

[10] Unlike most of the literature modeling bargaining directly in continuous time, we do not make any regularity/continuity/smoothness assumption on strategies to guarantee that standard recursive analysis can be used. Nevertheless, as we will see, equilibrium objects will be smooth enough that equation (6) will hold.

3.2 Equilibrium characterization

Equilibrium price

Fix an equilibrium for the rest of the section. From equations (4) and (5), and from the fact that the price at time T equals $p^*(\overline{v}_T)$ (i.e., the static monopolistic price when the valuation distribution is truncated above at $\overline{v}_T$), we obtain that

$$p(t,\overline{v}) = (1 - e^{-r_s\,(T-t)})\,c(\overline{v}) + e^{-r_s\,(T-t)}\,p^*(\overline{v}) \tag{7}$$

for any state $(t,\overline{v}) \in [0,T]\times(0,\overline{v}_0]$ (note that this expression is equivalent to expression (1)). The price at state $(t,\overline{v})$ is then a convex combination between the monopolistic price when the valuation is known to be below $\overline{v}$, $p^*(\overline{v})$, and the seller's valuation of the good if the buyer's valuation is $\overline{v}$, $c(\overline{v})$. As time comes close to the deadline, the seller commitment problem is reduced, and so the weight on the monopolistic price increases, and $p(T,\overline{v})=p^*(\overline{v})$.

Equation (7) provides a remarkably simple recipe to compute price in a given state $(t,\overline{v})$. First, compute the surplus the seller obtains from the buyer with valuation $\overline{v}$ in the static monopolist problem with valuations in $[0,\overline{v}]$ (which is equal to $p^*(\overline{v})-c(\overline{v})$). Second, "discount" the surplus using the remaining time until the deadline (at the seller's discount rate). This is equal to the surplus the seller obtains from the buyer with valuation $\overline{v}$ at state $(t,\overline{v})$; that is, $p(t,\overline{v})-c(\overline{v})$.

Note that Assumption 1 and the fact that F is differentiable imply that p^* is continuous and strictly increasing. Note also that $p(\cdot,\overline{v})$ is increasing if $p^*(\overline{v})>c(\overline{v})$: intuitively, the commitment problem of the seller becomes less severe as the deadline approaches, and she has more credibility on charging higher prices. Still, if $p^*(\overline{v})<c(\overline{v})$ we have that $p(\cdot,\overline{v})$ is a decreasing function: as t approaches T, the seller is more willing to make a loss when selling to the $\overline{v}$-buyer at time t to then obtain the monopolist payoff at the deadline. In both cases, the equilibrium price path $p_t=p(t,\overline{v}_t)$ decreases over time, as the decrease of $\overline{v}_t$ more than compensates the increase in t.

An important implication of equation (7) is that, for a given history p^t, there are no "trade bursts" on $[t,T)$. That is, there is no $t'\in[t,T)$ where $\overline{v}_{t'}(p^t)>\overline{v}_{t'^+}(p^t)$. To see this note that, by the optimality of the buyer's strategy and Proposition 3.1, the price $p_{t'}(t,\overline{v})$ is continuous in t' on $[t,T]$. Furthermore, the right-hand side of equation (7) is continuous and strictly increasing in $\overline{v}$. Then, the continuity of the on-path price $p_t=p(t,\overline{v}_t)$ implies that $\overline{v}_t$ is continuous too. The result is consistent with the finding in Fuchs and Skrzypacz (2013b) that the trade bursts obtained in the study of interdependent-value case (Deneckere and Liang, 2006) disappear in the limit where the gap between the lowest seller and buyer's valuations vanishes.

Equilibrium dynamics

We now use the optimality of the buyer's strategy to obtain the equilibrium price dynamics. For simplicity, we let p_t denote $P_t(\emptyset)$; that is, p_t is the price set by the seller at time t on the equilibrium path (the analysis of price dynamics after deviations is analogous). The "marginal buyer" at time $t \in (0, T)$ (i.e.; the buyer with valuation $\overline{v}_t$) is willing to purchase at time t, not before or after. Since $\overline{v}_t$ is continuous in time by the absence of trade bursts, the following equation holds:

$$\dot{p}_t = -r_{\mathrm{b}} \left(\overline{v}_t - p_t \right) . \tag{8}$$

The left-hand side of the previous expression is the instantaneous gain the $\overline{v}_t$-buyer obtains from delaying the purchase by an instant. The right-hand side is the cost owed to the delay of surplus he enjoys.

Equations (7) and (8) fully determine the equilibrium price dynamics. Indeed, we can use them to obtain

$$\overbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(c(\overline{v}_t)+e^{-r_{\mathrm{s}}(T-t)}\left(p^*(\overline{v}_t)-c(\overline{v}_t)\right)\right)}^{=\dot{p}_t} = -r_{\mathrm{b}}\left(\overline{v}_t-\Big(\overbrace{c(\overline{v}_t)+e^{-r_{\mathrm{s}}(T-t)}\left(p^*(\overline{v}_t)-c(\overline{v}_t)\right)}^{=p_t}\Big)\right) . \tag{9}$$

Equation (9) gives an ordinary differential equation (ODE) for the evolution of the upper threshold $\overline{v}_t$, and hence fully characterizes the equilibrium dynamics (note that the initial condition is that $\overline{v}_t$ at time 0 is equal to the parameter $\overline{v}_0$). The proof of Theorem 3.1 proves that the solution to equation (9) for $\overline{v}_t$ is, indeed, decreasing.

Main result

We gather the previous findings in the following result, characterizing the essentially unique equilibrium (i.e., generating a unique outcome after any state except for a zero-measure set of dates).

Theorem 3.1. *There is an essentially unique equilibrium. In such an equilibrium, after each state* $(t, \overline{v})$*,*

1. *the on-path threshold type* $\overline{v}_{t'}$ *solves equation* (9) *for all* $t' \in (t', T)$*, and is equal to* $\overline{v}$ *at time* t*;*
2. *the price is equal to* $p(t', \overline{v}_{t'})$ *given in equation* (7) *for almost all* $t' \in (t', T)$*, and*
3. *the buyer with valuation* $v \in [0, \overline{v}]$ *buys at the time* t^v *given by (recall that* $+\infty$ *means "never buys"):*

$$t^v = \begin{cases} t \text{ solving } \overline{v}_t = v & \text{if } v \in (\overline{v}_T, \overline{v}], \\ T & \text{if } v \in [p^*(\overline{v}_T), \overline{v}_T], \\ +\infty & \text{otherwise.} \end{cases}$$

Consider now the limit $T \to 0$. Theorem 3.1 establishes that, as T gets small, (i) the seller's payoff converges to the static monopolistic profits $\Pi^*(\overline{v}_0)$, (ii) the initial price converges to $p^*(\overline{v}_0)$, and (iii), in equilibrium, the seller remains willing to screen infinitely fast. Hence, an almost direct implication of the theorem is that the static monopolistic profits are equal to the payoff that the seller would get in a market if she was able to perfectly price-discriminate by charging, to each type of the buyer $\overline{v}$, the monopolistic price she would charge if it was known that the buyer's valuation was lower than $\overline{v}$ (i.e., $p^*(\overline{v})$). We prove this implication independently, using the envelope theorem:

Corollary 3.1. *We have* $\Pi^*(\overline{v}_0) = \int_0^{\overline{v}_0} (p^*(v) - c(v))\, F(\mathrm{d}v)$.

3.3 Discussion of Assumption 2

Assumption 2 plays a critical role in the proof of Proposition 3.1. The intuitive argument outlined in Section 3.1 proceeds by contradiction by assuming there is no trade in some interval (t_1, t_2). It is shown that Assumption 2 guarantees that, even if the price at t_2 is 0, the seller gains from deviating and selling earlier to the buyer with a high valuation.

Proposition 3.1 is then used to prove Propositions 3.2 and 3.3. As a result, the equilibrium price satisfies equation (7) when Assumption 2 holds. Hence, the cost-benefit argument used to show Proposition 3.1 can be replicated using the equilibrium price instead of 0. Doing so permits obtaining a condition which is less restrictive than Assumption 2 and which is sufficient for the "no silent period" condition to hold.

Assumption 3. For any $\overline{v} \in (0, \overline{v}_0]$, we have $r_{\mathrm{b}}\,(\overline{v} - p^*(\overline{v})) \geq r_{\mathrm{s}}\,(c(\overline{v}) - p^*(\overline{v}))$.

It is easy to see that Assumption 3 is less restrictive than Assumption 2 (see the proof of Theorem 3.1). In fact, Assumption 2 is the least restrictive condition that guarantees that Assumption 3 holds *independently* of the distribution of buyer's valuations. That is, if Assumption 2 holds then Assumption 3 holds as well (independently of F), and for any $(c, r_{\mathrm{s}}, r_{\mathrm{b}})$ not satisfying Assumption 2 there is some distribution F (and corresponding p^*) for which Assumption 3 does not hold. Additionally, the proof of Theorem 3.1 shows that Assumption 3 is necessary and sufficient for the solution of equation (9) for $\overline{v}_t$ to be decreasing. Hence, we have:

Corollary 3.2. *The strategy profile described in Theorem 3.1 is an equilibrium if and only if Assumption 3 holds.*

Scope

We say that *adverse selection is not strong* if $p^*(\overline{v}) \geq c(\overline{v})$ for all $\overline{v} \in (0, \overline{v}_0]$; that is, if for any $\overline{v}$ the static monopolist payoff is non-negative valuation-by-valuation. Requiring adverse selection not to be strong is a sufficient condition for Assumption 3 to hold and, in fact, it is the broadest condition that guarantees that the strategy profile described in Theorem 3.1 is an equilibrium *independently* of the values of r_{s} and r_{b}.

When does then Assumption 3 *not* hold? That is, when is the strategy profile described in Theorem 3.1 *not* an equilibrium? For this to occur, it is necessary that (i) Assumption 2 does not hold,[11] and (ii) the seller's valuation is above the monopolistic price for some of the buyer's valuations. This is the case if, for example, the seller is more impatient and she has a high valuation for the good with a small probability. If this occurs, the static monopolist price $p^*(\overline{v})$ is low even for large values of $\overline{v}$, and so $c(\overline{v}_0) > p^*(\overline{v}_0)$. Then, if the seller is impatient enough, she will not be willing to sell at time 0 at a price which, by equation (7), is lower than $c(\overline{v}_0)$ (see Figure 4 below).

4 Comparative statics and other results

4.1 Buyer's patience

We first investigate how changes in the buyer's patience affect the outcome of the game.

From Proposition 3.2 it is clear that the payoff of the seller is not affected by the value of r_{b}, and neither is the price at time 0 by equation (7). The price path and the timing of purchases, however, do depend on the buyer's patience level. The following result establishes that, if adverse selection is not strong, the buyer benefits from facing a higher interest rate. In other words, he is more willing to pay to enter the market when his bargaining cost is higher.

Proposition 4.1. *The seller's payoff is independent of r_{b}. If adverse selection is not strong then, for any $v \in [0, \overline{v}_0]$, the v-buyer's payoff is increasing in r_{b}.*

An intuition for Proposition 4.1 is the following. An increase in the buyer's discount factor does not affect the commitment problem of the seller, and hence it does not change the equilibrium price in each state $(t, \overline{v})$ (see equation (7)) or her payoff (see equation (5)). The speed at which the price declines, nevertheless, increases: since the buyer becomes more impatient, he remains indifferent from buying now or an instant after only if the price declines faster (see equation (8)). In other words, rejecting a given price offer becomes a stronger signal of a low valuation when the buyer is more impatient, which forces the seller to lower the price faster. Therefore, the buyer is screened more rapidly. Importantly, the faster price decline is reinforced, in equilibrium, by the fact that each given $\overline{v}$ is reached at an earlier time, and hence this type of the buyer pays a lower price. Hence, again by equation (8), the speed at which price declines at the instant where each type of the buyer $\overline{v}$ buys increases more than proportionally than the increase in r_{b}. Such reinforcement gives the result: the additional impatience is more than compensated, in equilibrium, by the more-than-proportional of the speed at which the price declines.

[11] Recall that, as we explain above, Assumption 2 holds in two canonical cases: the private values case, and the case where the seller is (weakly) more patient than the buyer.

When adverse selection is strong, the effect of an increase in the buyer's impatience is ambiguous. From the proof of Proposition 4.1 it is easily seen that, if for example $p^*(\overline{v}_0) < c(\overline{v}_0)$, a buyer with a high valuation gets worse off when the interest rate he faces increases. Still, similarly to the example presented in the next section, a buyer with a lower valuation may become better off because of the fast screening of low valuations.

4.2 Seller's patience

We now present the comparative statics analysis with respect to the seller's discount rate r_s. From Proposition 3.3 we have that the seller's payoff is decreasing in the interest rate he faces, r_s. The change in the payoff of the buyer depends on whether adverse selection is strong or not.

When adverse selection is not strong, the payoff of the buyer is increasing in r_s independently of his valuation. This follows from the observation that the price in each given state $(t, \overline{v})$ (see equation (7)) is decreasing in r_s, and hence —using an argument similar to that in the proof of Proposition 4.1— the price decreases faster when r_s is larger. Both results are easily anticipated: a higher discount rate exacerbates the seller's commitment problem. We formalize these claims in a proposition:

Proposition 4.2. *The seller's payoff is decreasing in r_s. If adverse selection is not strong then, for any $v \in [0, \overline{v}_0]$, the v-buyer's payoff is increasing in r_s.*

Assume alternatively that adverse selection is strong and, in particular, that $p^*(\overline{v}_0) < c(\overline{v}_0)$. In this case, a more impatient seller charges higher initial prices. The reason is that a seller who faces a higher interest rate is less willing to make losses at earlier dates to get a larger payoff at later times. For lower buyer valuations, instead, an increase in the seller's interest rate decreases the price at each state, and accelerates the price decline. Hence, while buyers with a high valuation prefer a more patient seller, buyers with a low valuation may prefer a less patient seller.

Figure 4 depicts an example where adverse selection is strong. Subfigure (a) shows that, for high valuations, the static monopolist price is lower than the cost. Intuitively, the seller is willing to make a loss from selling to buyers with a high valuation in order to obtain a large payoff from selling to buyers with an intermediate valuation. Subfigure (b) depicts the price and threshold types trajectories for small r_s (gray lines) and large r_s (black lines). When the seller is more impatient, initial prices are higher, while later prices are lower. Intuitively, a more impatient seller is less willing to make losses at the beginning (by selling to high-valuation buyers), as the larger discounting makes it more difficult to compensate them with the posterior sales to buyers with a middle valuation. Later, when the cost is below the static monopolist price, the large impatience implies faster screening. This intuition is confirmed in subfigure (c): when the seller is more impatient, the initial flow payoff is less negative, but the later flow payoff is smaller. Overall, we see that a more impatient seller initially screens high-cost buyers slower by charging

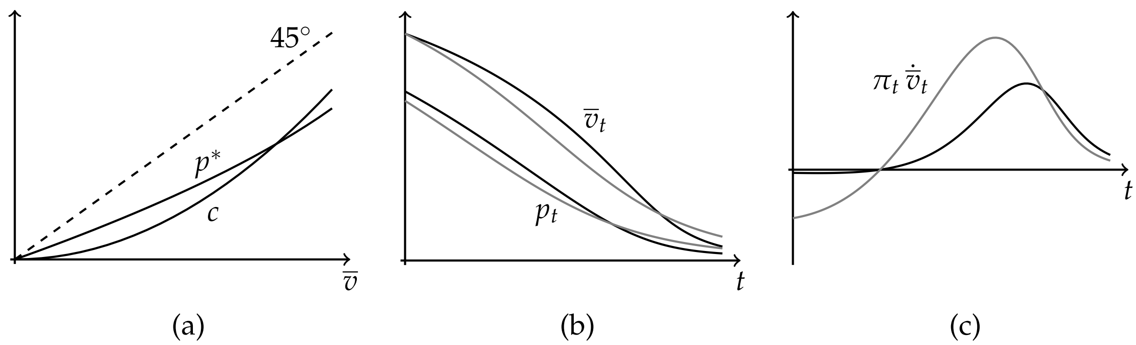

Figure 4: For $T=3$, $r_{\mathrm{b}}=1$, F uniform in $[0,1]$, $c(v)=\frac{3}{4}\,v^2$, and for $r_{\mathrm{s}}=0.1$ (gray lines) and $r_{\mathrm{s}}=1$ (black lines), (a) depicts $p^*(\overline{v})$ and $c(\overline{v})$, (b) depicts threshold types $\overline{v}_t$ (upper lines) and prices p_t (lower lines), and (c) depicts the flow payoff of the seller, $\pi_t\,\dot{\overline{v}}_t \equiv (p_t - c(\overline{v}_t))\,\dot{\overline{v}}_t$.

a higher price and, later, buyers with a medium or a low valuation are screened faster.

4.3 Time horizon

The effect of increasing the deadline T is similar to that of increasing the seller's interest rate r_{s}.

Proposition 4.3. *The seller's payoff is decreasing in T. If adverse selection is not strong then, for any $v \in [0,\overline{v}_0]$, the v-buyer's payoff is increasing in T.*

The seller's payoff is decreasing in T, as the commitment problem of the seller worsens when she has more time to screen the buyer. If adverse selection is not strong, raising the deadline from T to $T' > T$ implies that the equilibrium price at state $(T,\overline{v})$ becomes lower. This translates, in equilibrium, into lower initial prices, and so the buyer is better off. If adverse selection is strong, instead, extending the deadline translates into higher initial prices. The intuition is similar to the effect of increasing the seller's interest rate: the increased commitment problem of the seller makes sales in later dates less profitable, inducing the seller to charge higher initial prices.

An interesting exercise consists in considering simultaneous changes of r_{s} and T that keep $r_{\mathrm{s}}\,T$ constant. These changes do not affect the equilibrium commitment problem of the seller: her payoff depends on r_{s} and T only through $r_{\mathrm{s}}\,T$. The payoff of the buyer is, nevertheless, affected by such changes. An argument similar to that in the proof of Proposition 4.1 illustrates that such changes are beneficial for the buyer when the seller becomes more patient (i.e., when T increases and r_{s} decreases while keeping $r_{\mathrm{s}}\,T$ the same) when adverse selection is not strong.

Infinite horizon limit

We now analyze the limit where the time horizon becomes large. As T increases, our model approximates continuous-time versions of Gul, Sonnenschein, and Wilson (1986) and Deneckere and Liang

(2006).

If $c \equiv 0$ (private values case) then, by equation (7), $p(t, \overline{v}) \to 0$ as $T \to \infty$ for all $(t, \overline{v})$. In other words, we recover the Coase conjecture. Even if the seller is more patient than the buyer, the inability to commit not to lower future prices dissipates all her rents from trade.

When $c(\cdot)$ is strictly increasing, we can combine of equations (7) and (8) gives

$$\tfrac{\mathrm{d}}{\mathrm{d}t} c(\overline{v}_t) = -r_{\mathrm{b}} \left(\overline{v}_t - c(\overline{v}_t) \right) . \tag{10}$$

This is the same equation obtained in Fuchs and Skrzypacz (2013b) for the case $r_{\mathrm{s}} = r_{\mathrm{b}}$ in the double-limit where the gap between the lowest seller's and buyer's valuations and the length of the period vanish. We obtain that not only price dynamics are independent of the distribution of buyer's valuations in the infinite-horizon model (as observed by Fuchs and Skrzypacz), but they are also independent of the seller's patience level. Furthermore, for each value v, payoff of the buyer with valuation v is independent of r_{s}, r_{b}, or F.[12]

Note that the dynamics described in Theorem 3.1 in the limit where $T \to \infty$ coincide with the equilibrium dynamics in the limit $r_{\mathrm{s}} \to \infty$ when adverse selection is not strong (and hence Assumption 3 holds for all r_{s}): in both cases $p(t, \overline{v}) = c(\overline{v})$ for all $(t, \overline{v})$ and the seller's payoff is zero. The limit outcome in the double limit $r_{\mathrm{s}}, T \to \infty$ can be re-interpreted as the outcome of a model with an infinite sequence of short-lived sellers, studied in Hörner and Vieille (2009) (public offers case). Our results have the implication that the "trade impasse" disappears (and trade is smooth) when there is no gap between the lowest valuations of the seller and buyer and adverse selection is not strong. In fact, differently from them, we find that there is no trade burst at time 0, and trade occurs smoothly and evenutally with probability one. This result can be seen as analogous to that in Fuchs and Skrzypacz (2013b): they show that the trade bursts predicted by Deneckere and Liang (2006) (for the long-lived seller with the same discount as the buyer) do not occur in the no-gap case.

4.4 Commitment case for the uniform distribution and linear costs

The seller's inability to commit not to lower the price implies that her payoff is strictly lower than the one she would obtain if she could commit. This is clear when $r_{\mathrm{s}} \geq r_{\mathrm{b}}$: Stokey (1979) showed that, in this case, trade occurs only at time 0 and at the monopolistic price $p^*(\overline{v}_0)$ when the seller can commit (the result can be extended to the interdependent values case).

When $r_{\mathrm{s}} < r_{\mathrm{b}}$, a seller with commitment price-discriminates, taking advantage of the larger delay cost of the buyer (see Fudenberg and Tirole, 1983, and Landsberger and Meilijson, 1985).[13] The com-

[12]Solving equation (10) we have that the payoff of the v-buyer is $\exp\left(-\int_{v}^{\overline{v}_0} \frac{c'(v')}{v' - c(v')} \mathrm{d}v'\right)(v - c(v))$.

[13]Beccuti and Möller (2018) analyze a two-type, discrete-time, private-values model where the seller can offer a

mitment solution is, in general, difficult to obtain. To gather some intuition, we heuristically derive the optimal (non-stochastic) pricing strategy of a seller with commitment when the buyer's valuation is distributed uniformly on $[0,1]$ and the seller's cost is linear, $c(v) = k\,v$ for some $k \in [0,1)$. In this case $p^*(\overline{v}) = \frac{1}{2-k}\,\overline{v}$. We also focus, for simplicity, in the case where the seller is fully patient; i.e., $r_{\mathrm{s}} = 0$.

In order to apply calculus of variations, we assume equation (8) holds in any interior time interval (t_1, t_2). Hence, the objective function of the seller in this interval is

$$\int_{t_1}^{t_2} (p_t - c(\overline{v}_t))\,\dot{\overline{v}}_t\,\mathrm{d}t = \int_{t_1}^{t_2} \underbrace{\left(p_t - k\,(p_t - r_{\mathrm{b}}^{-1}\,\dot{p}_t)\right)(\dot{p}_t - r_{\mathrm{b}}^{-1}\,\ddot{p}_t)}_{\equiv L(p_t,\dot{p}_t,\ddot{p}_t)}\,\mathrm{d}t\;.$$

Standard calculus of variations requires that the Euler–Lagrange equation holds:

$$0 = \frac{\partial L}{\partial p_t} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{p}_t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{p}_t} = -\frac{2\,\ddot{p}_t}{r_{\mathrm{b}}}\;.$$

That is, the price is a linear function time. The seller's problem is then bound to find the optimal values for the prices at dates 0 and T, denoted p_0 and p_T, respectively. Equation (8) dictates that $\overline{v}(p) \equiv p - \frac{p_T - p_0}{r_{\mathrm{b}}\,T}$, and hence the seller chooses p_0 and p_T to maximize

$$(1 - \overline{v}(p_0))\,p_0 + (\overline{v}(p_0) - \overline{v}(p_T))\,\tfrac{p_T + p_0}{2} + (\overline{v}(p_T) - p_T)\,p_T - \tfrac{k}{2}\,(1 - p_T^2)\;.$$

There is a unique pair $\{p_0^{\mathrm{c}}, p_T^{\mathrm{c}}\}$ maximizing the previous expression. We can use these values to obtain the price the committed seller charges:

$$p_t^{\mathrm{c}} \equiv \frac{2 - k + (1-k)\,r_{\mathrm{b}}\,(T - t)}{2 - k + (1-k)\,r_{\mathrm{b}}\,T}\;.$$

From equations (7) and (8) we have that, in the absence of seller's commitment, the equilibrium price is $p_t^{\mathrm{nc}} = \frac{1}{2-k}\,e^{-r_{\mathrm{b}}\,(1-k)\,t}$. Figure 5 depicts the price and upper valuation paths for the commitment and no-commitment cases for $k = 0$ (private values case) and $k = \frac{1}{2}$ (interdependent values case).

There are three observations worth mentioning. First, since the seller is fully patient, her equilibrium payoff is equal to the static monopolistic payoff $\Pi^*(\overline{v}_0)$. The equilibrium price, nevertheless, decreases over time. Hence, even though the seller price-discriminates both when she can and when she can not commit, she fails to benefit from such price discrimination when she does not have commitment power. Second, in line with what occurs in the non-commitment case, the price at the deadline coincides with the static monopolistic price for the remaining buyer types. Hence, the seller's commitment problem does not arise from the inability to commit not to lower the price at the very end (and using then a

mechanism in each period and is more patient than the buyer. They obtain significant differences between selling and renting mechanisms.

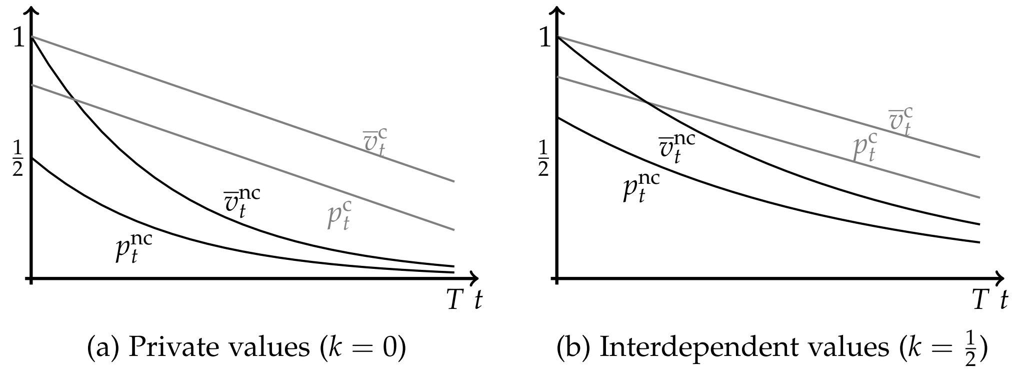

Figure 5: Price and upper valuation in the commitment (gray lines) and non-commitment (black lines) solutions, for F uniform on $[0,1]$, $r_{\mathrm{s}}=0$, $r_{\mathrm{b}}=1$, $T=3$, and $c(v)=k\,v$. For all cases, the only trade burst occurs at the deadline. As expected, in the non-commitment case, there is a higher trade probability and a lower trade delay in comparison to the commitment case (i.e., $\overline{v}_t^{\mathrm{nc}}<\overline{v}_t^{\mathrm{c}}$ for all $t>0$). Also, there is a higher probability of trade and less trade delay in the private values case.

backward induction argument), but from the inability not to lower the price before. Finally, in the commitment solution, every valuation of the buyer trades at a later date (or not at all) compared with the non-commitment solution. Hence, opposite to the case where the seller is more impatient than the buyer (where trade occurs immediately when the seller has commitment power), giving commitment power to a patient seller not only decreases the probability of trade, but also increases the trade delay.

4.5 Time-dependent discount rates

In this section we study the case where the cost of bargaining is time-dependent. In practice, increasing discount rates may correspond to an increasing probability of exogenous breakdown, a stochastic value decline (see Hart, 1989), or changes in the idyosincratic interest rate.

We now consider the case where the discount rates of the seller and the receiver are respectively given by two bounded functions $r_{\mathrm{s}}, r_{\mathrm{b}} : [0,T] \to \mathbb{R}_{++}$. Then, we now analyze the same model as in Section 2, the only difference being that if transaction occurs at time t at price p_t, the payoffs or the seller and the buyer are, respectively,

$$e^{-\int_0^t r_{\mathrm{s}}(t)\,\mathrm{d}t}\,(p_t - c(v)) \;\text{ and }\; e^{-\int_0^t r_{\mathrm{b}}(t)\,\mathrm{d}t}\,(v - p_t)\,.$$

Now, Assumption 2 is replaced by requiring that, for any $\overline{v}\in(0,\overline{v}_0]$ and t, $c(\overline{v})\leq \frac{r_{\mathrm{b}}(t)}{r_{\mathrm{s}}(t)}\,\overline{v}$.

It is not difficult to see that all results in Sections 2.1 and 3.1 still hold (changing equation (5) accordingly). This can be best seen by normalizing the time unit so that the seller has a constant discount factor, for example equal to 1. Then, most arguments in the proofs of the results follow immediately. In this normalized model, equation (7) holds with $r_{\mathrm{s}}=1$. The normalized, time-dependent discount rate for the buyer, $\tilde{r}_{\mathrm{b}}(t)$, modulates the speed at which the price changes through equation (8).

Consider first the case where $r_{\rm s}(t)$ and $r_{\rm b}(t)$ increase over time, but $r_{\rm b}(t)/r_{\rm s}(t)$ is constant and equal to κ, as in Hart (1989).[14] The equilibrium can be found by first defining $\hat{r}_{\rm s} \equiv \int_0^T r_{\rm s}(t)\,{\rm d}t/T$. Let $\hat{p}_t$ be on-path equilibrium price for the model with seller's discount rate $\hat{r}_{\rm s}$, buyer's discount rate $\kappa\,\hat{r}_{\rm s}$, and time horizon T. The price in the model with time-dependent discount rates is then given by:

$$p_t = \hat{p}_{\hat{t}(t)}\ , \quad \text{where } \hat{t}(t) \equiv \hat{r}_{\rm s}^{-1} \int_0^t r_{\rm s}(t)\,{\rm d}t\ .$$

Since $r_{\rm s}(\cdot)$ is increasing, $\hat{t}(\cdot)$ is convex. Hence, p_t is an "accelerated" version of the price in an analogous model with constant discount rates: as delay becomes more costly, the probability of agreement increases. The payoffs of the seller and the buyer coincide with their payoffs in the normalized model.

To gain further intuition, consider the private-values case and general $r_{\rm s}(\cdot)$ and $r_{\rm b}(\cdot)$. We can then differentiate $p_t = p(t, \overline{v}_t)$ using equation (7), and we obtain

$$\dot{p}_t = e^{-\int_t^T r_{\rm s}(t')\,{\rm d}t'}\,\big(\underbrace{r_{\rm s}(t)\,p^*(\overline{v}_t)}_{(*)} + \underbrace{p^{*\prime}(\overline{v}_t)\,\dot{\overline{v}}_t}_{(**)}\big) \tag{11}$$

If the value of $r_{\rm s}(\cdot)$ becomes larger for a given interval of time while $r_{\rm b}(\cdot)$ remains roughly the same, the price $p(t, \overline{v}_t)$ cannot decrease much faster in this interval (by equation (8)). Given that the term $(*)$ in equation (11) is larger, the term $(**)$ has to be more negative, and the seller screens the buyer faster. Conversely, if the value of $r_{\rm b}(\cdot)$ increases for a given interval of time while $r_{\rm s}(\cdot)$ remains roughly the same, the price decreases fast in this interval (by equation (8)). Hence, the term $(*)$ in equation (11) does not change much, and so the term $(**)$ becomes more negative; as a result, the buyer is screened faster. Summarizing, while both periods of large seller discounting and periods of large buyer discounting translate into faster buyer screening, only periods of large buyer discounting translate into faster price decline.

5 Conclusions

Delay and failure in reaching an agreement are commonly observed in real-life bargaining. Gaining understanding of the outcome of negotiations requires determining the role of different factors, such as the cost of bargaining, the gains from trade, or the deadlines that govern the bargaining process.

The tractability of our model permits characterizing how different factors affect the outcome of negotiations with private information. While some of our results validate standard intuitions, some others turn out to be less straightforward. Our analysis identifies the conditions on the gains from trade

[14] Hart (1989) studies a two-type model in discrete time where, in the period before a deadline, the probability of decline of the value significantly increases. He finds that most trade occurs in the first and the last periods.

that determine the qualitative effects that changes in delay costs have on the trade outcome. These effects depend on whether adverse selection is strong or not; that is, on whether the cost of supplying high-cost buyers is above the static monopolistic price or not.

When adverse selection is not strong, the buyer is better off when his delay cost is higher. Intuitively, a higher buyer interest rate implies that the buyer is less willing to reject each given price offer, lowering the endogenous valuation conditional on having rejected the previous offers. This induces the seller to decrease prices faster. When the horizon is finite, there is an additional effect: the seller's commitment problem is more severe at earlier dates. The additional effect accelerates the price decline further, implying that the buyer is better off even though his delay cost is higher.

When adverse selection is strong, the seller's payoff from initial sales is negative. In equilibrium, initial losses are compensated by more profitable sales in later dates. When the seller faces a higher interest rate, she is less willing to intertemporally trade-off losses and gains; hence, a more impatient seller charges higher initial prices and delays trade. A buyer with a high valuation is then worse off when the seller is more impatient, while a buyer with a low valuation benefits from the fast price decline in later dates. A similar intuition applies when the deadline becomes longer: the seller's increased commitment problem makes sales in later dates less profitable, inducing her to charge higher initial prices to avoid losses.

Future research can be devoted to extend our results beyond some of our assumptions. For example, as we argue in Section 3.3, our constructed equilibrium fails to be an equilibrium when the seller is very impatient and adverse selection is strong. In this case, there may be trade impasses; that is, intervals of time without trade. Similarly, the analysis could be extended to the gap case, where the buyer valuation is bounded away from 0.[15] This could result in equilibria where agreement is reached before the deadline for sure. Nevertheless, note that our assumptions on the distribution permit approximating the gap case through distributions with assigning increasingly lower probability to lower valuations.

[15] In discrete time, the finite horizon plays a similar role as the gap. In particular, the folk theorem in Ausubel and Deneckere (1989) for the no-gap case fails when the horizon is finite, as one can use backward induction from the last period with trade.

A Proofs of the results

A.1 Proofs of results in Section 2

Proof of Lemma 2.1

Proof. The proof follows the standard argument. Fix some PBE (P, a, F) and history p^t. Take two valuations $v > v'$ and assume, for the sake of contradiction, that $t^v > t^{v'}$ (we omit ther explicit dependence on the strategy profile and the history). Note that the optimality of the buyer's strategy requires that $p^v < p^{v'} < v'$. Then, we have

$$\begin{aligned}
\exp\left(-r_{\mathrm{b}}\, t^v\right)(v - v') &= \exp\left(-r_{\mathrm{b}}\, t^v\right)(v - p^v) - \exp\left(-r_{\mathrm{b}}\, t^v\right)(v' - p^v) \\
&\geq \exp\left(-r_{\mathrm{b}}\, t^{v'}\right)(v - p^{v'}) - \exp\left(-r_{\mathrm{b}}\, t^{v'}\right)(v' - p^{v'}) \\
&= \exp\left(-r_{\mathrm{b}}\, t^{v'}\right)(v - v') \;,
\end{aligned}$$

which contradicts that $t^v > t^{v'}$. The inequality holds because the v-buyer is (weakly) worse off following the v'-buyer's equilibrium strategy and the v'-buyer is (weakly) better off following his equilibrium strategy. □

A.2 Proofs of results in Section 3

Proof of Proposition 3.1

Proof. We begin the proof with an auxiliary result:

Lemma A.1. *Let p^t and $\hat{p}^t$ be such that $\overline{v}(p^t) < \overline{v}(\hat{p}^t)$. Then*

$$\Pi(t, \overline{v}(p^t)) \geq \tfrac{1}{F(\overline{v}(p^t))} \int_0^{\overline{v}} e^{-r_{\mathrm{s}}\,(t^v(\hat{p}^t; P, a) - t)} \left(p^v(\hat{p}^t; P, a) - c(v)\right) F(\mathrm{d}v) \;. \tag{12}$$

Proof. Let $\hat{P}$ be a seller's strategy defined by $\hat{P}(p^{t'}) \equiv P(P^{t'}(\hat{p}^t))$ for all $p^{t'}$. Intuitively, for each t', strategy $\hat{P}$ requires the seller to do what strategy P specifies after history $P^{t'}(\hat{p}^t)$. Then, from the second condition in Definition 2.2, we have the seller's payoff from using strategy $\hat{P}$ after history p^t coincides with the right-hand side of equation (12). Hence, the seller's payoff at state $(t, \overline{v}(p^t))$ can not be lower than the right-hand side of equation (12) □

(Proof of Proposition 3.1 continues.)

The proof of Lemma A.1 uses a mimicking argument. The seller at state $(t, \overline{v})$ can imitate the continuation strategy after state $(t, \overline{v}')$ with $\overline{v}' > \overline{v}$. By doing this, the seller's payoff from buyer types below $\overline{v}$ coincides with the one she obtains from these types after state $(t, \overline{v}')$.

Fix an equilibrium and history p^{t_1}, for some $t_1 \in [0, T)$. Assume, for the sake of contradiction, that there is $t_2 > t_1$ satisfying that $\overline{v}_{t_1} = \overline{v}_{t_2^-} > 0$ (as noted before, the explicit dependence on the history p^{t_1} is omitted).[16] We first observe that it must be that $\overline{v}_T < \overline{v}_{t_1}$, since $\overline{v}_{T^-} \leq \overline{v}_{t_2^-}$ (by the skimming property) and $\overline{v}_T = p^*(\overline{v}_{T^-})$. It is convenient to pick t_2 to be the supremum among the times in $(t_1, T]$ satisfying that $\overline{v}_{t_2^-} = \overline{v}_{t_1}$. We consider two separate cases:

Case 1: There is a "burst" of trade at time t_2. Assume first $\overline{v}_{t_2} < \overline{v}_{t_2^-}$; that is, there is a positive probability of trade at time t_2 (at price $P_{t_2} \equiv P_{t_2}(p^{t_1})$). Fix some $t \in (t_1, t_2)$ and some $\varepsilon > 0$, and consider the following deviation by the seller at time t: offer price

$$\hat{p}_t \equiv (1 - e^{-r_{\mathrm{b}}(t_2 - t)})\,(\overline{v}_{t_1} - \varepsilon) + e^{-r_{\mathrm{b}}(t_2 - t)}\,P_{t_2} \tag{13}$$

at date t, unacceptable prices in (t, t_2), and continue with the equilibrium strategy $P(p^{t_1})$ from date t_2 on (that is, "as if" she did not deviate at time t). Note that the $(\overline{v}_{t_1} - \varepsilon)$-buyer is indifferent between accepting $\hat{p}_t$ at time t and accepting P_{t_2} at time t_2. Hence, for all $v \in (\overline{v}_{t_1} - \varepsilon, \overline{v}_{t_1})$ the v-buyer obtains a strictly bigger payoff from accepting $\hat{p}_t$ at time t than from accepting P_{t_2} at time t_2. Let $\hat{v}_t$ denote $\overline{v}(P^{t^-}(p^{t_1}), \hat{p}_t)$.

There are two possibilities. The first is that there is no trade at time t when $\hat{p}_t$ is offered; that is, $\hat{v}_t = \overline{v}_{t_1}$. Then, by the Markov property, the buyer believes that the seller's continuation strategy is such that there is no trade until t_2, where the price is P_{t_2}. This, nevertheless, leads to a contradiction, since as we observed before, there is a positive mass of buyer's valuations such that the buyer strictly prefers accepting $\hat{p}_t$ at t to accepting P_{t_2} at t_2. The second possibility is that there is a positive probability of trade at date t when $\hat{p}_t$ is offered; that is, $\hat{v}_t < \overline{v}_{t_1}$. By the argument in the proof of Lemma A.1, the seller obtains the same payoff from all types $v < \hat{v}_t$ under the deviation than under the equilibrium strategy (given that, under our deviation, her continuation strategy coincides with the continuation strategy if she did not deviate). The increase in the seller's payoff from the buyer when his valuation is in $[\hat{v}_t, \overline{v}_{t_1}]$ is given by

$$\overbrace{\hat{p}_t - \mathbb{E}[c(\tilde{v})|\tilde{v} \in [\hat{v}_t, \overline{v}_{t_1}]]}^{\text{payoff when } \hat{p}_t \text{ is offered}} - \overbrace{e^{-r_{\mathrm{s}}(t_2 - t)}\,(P_{t_2} - \mathbb{E}[c(\tilde{v})|\tilde{v} \in [\hat{v}_t, \overline{v}_{t_1}]])}^{\text{equilibrium payoff}}\,.$$

Using equation (13) we have that, as $t_2 - t \to 0$, the previous expression can be written as:

$$\Big(\underbrace{(\overline{v}_{t_1} - \varepsilon - P_{t_2}) - r_{\mathrm{s}}\,(\mathbb{E}[c(\tilde{v})|\tilde{v} \in [\hat{v}_t, \overline{v}_{t_1}]] - P_{t_2})}_{(*)}\Big)\,(t_2 - t) + O((t_2 - t)^2)\,.$$

Note that, since $c(\cdot)$ is increasing, $\mathbb{E}[c(\tilde{v})|\tilde{v} \in [\hat{v}_t, \overline{v}_{t_1}]] \leq c(\overline{v}_{t_1})$. Note also that the term $(*)$ is linear in

[16]Note that, for each t_1, there exists $t_2 > t_1$ such that $\overline{v}_{t_1} = \overline{v}_{t_2^-}$ if and only if there exists $t_2' > t_2$ such that $\overline{v}_{t_1} = \overline{v}_{t_2'}$.

P_{t_2}, larger than $r_{\rm s}\,(\overline{v}_{t_1}-c(\overline{v}_{t_1}))-r_{\rm b}\,\varepsilon$ when $P_{t_2}=\overline{v}_{t_1}$, and larger than $r_{\rm b}\left(\overline{v}_{t_1}-\frac{r_{\rm s}}{r_{\rm b}}\,c(\overline{v}_{t_1})-\varepsilon\right)$ when $P_{t_2}=0$. Hence, using that $\overline{v}_{t_1}>c(\overline{v}_{t_1})$ and Assumption 2, we have that $(*)$ is postitive if ε is small enough. We then conclude that there exists a profitable deviation for the seller, a contradiction.

Case 2: There is no "burst" of trade at time t_2. Assume now $\overline{v}_{t_2^-}=\overline{v}_{t_2}$. The logic for this case is similar for Case 1, but the argument slightly more involving. First note that, by the observation above, it must be that $t_2<T$. Pick again some $t\in(t_1,t_2)$ and now let $\tilde{p}_t$ be such that the $(\overline{v}_{t_2}-\varepsilon)$-buyer is indifferent between accepting $\tilde{p}_t$ at time t or $p^{\overline{v}_{t_2}-\varepsilon}$ at time $t^{\overline{v}_{t_2}-\varepsilon}$. Noticing that, if ε is small enough, $t^{\overline{v}_{t_2}-\varepsilon}$ is close to t_2 (by the definition of t_2), the same argument as in Case 1 goes through. □

Proof of Proposition 3.2

Proof. The proof is divided into three lemmas. Lemma A.2 sets an upper bound on the seller's payoff. Lemma A.3 establishes a lower and an upper bound on the seller's payoff in terms of the continuation payoffs at lower threshold valuations. Lemma A.4 establishes the continuity of $p(t,\cdot)$ for all t. We finally argue that these lemmas imply the result stated in Proposition 3.2.

We first present an auxiliary result. The result establishes that, after any t-history, the seller's payoff is no higher than tha from selling to higher buyer types at the price at time t, while selling at the same time and prices to the lower buyer types. The result is intuitive, as the seller sells earlier and at a higher price to higher types. Nevertheless, the fact that the seller incurs the cost of selling to the higher types at an earlier time makes the result not trivial.

Lemma A.2. *Fix some state $(t,\overline{v})$. Then, for all $\overline{v}'\leq\overline{v}$, we have*

$$\Pi(t,\overline{v})\leq\tfrac{F(\overline{v})-F(\overline{v}')}{F(\overline{v})}\left(p(t,\overline{v})-\mathbb{E}[c(\tilde{v})|\tilde{v}\in[\overline{v}',\overline{v}]]\right)+\tfrac{F(\overline{v}')}{F(\overline{v})}\int_0^{\overline{v}'}e^{-r_{\rm s}\,(t^v-t)}\,\pi_{t^v}(t,\overline{v})\,F(\mathrm{d}v)\;. \tag{14}$$

Proof. To prove the result, we show that selling to each of the types $v\in(\overline{v}',\overline{v})$ at time t at price $p(t,\overline{v})$ gives the seller a payoff which is larger than selling to them at time $t^v(t,\overline{v})$ at price $p_{t^v}(t,\overline{v})$. By the optimality of the buyer's strategy, it must be that

$$v-p(t,\overline{v})\leq e^{-r_{\rm b}\,(t^v(t,\overline{v})-t)}\,(v-p_{t^v}(t,\overline{v}))\;.$$

Then, from the previous inequality, we have that the payoff the seller obtains from selling to the buyer with valuation v at time t at price $p(t,\overline{v})$ —which is equal to $p(t,\overline{v})-c(v)$— is no smaller than

$$\begin{aligned}&e^{-r_{\rm s}\,(t^v(t,\overline{v})-t)}\,(p_{t^v}(t,\overline{v})-c(v))\\&\quad+\underbrace{(e^{-r_{\rm b}\,(t^v(t,\overline{v})-t)}-e^{-r_{\rm s}\,(t^v(t,\overline{v})-t)})\,p(t,\overline{v})+(1-e^{-r_{\rm b}\,(t^v(t,\overline{v})-t)})\,v-(1-e^{-r_{\rm s}\,(t^v(t,\overline{v})-t)})\,c(v)}_{(*)}\;.\end{aligned}$$

It is only left to prove that the term $(*)$ is bigger than 0. Note that $(*)$ is linear in $p(t,\overline{v})$, and it is equal to $(1-e^{-r_{\rm s}\,(t^{v}(t,\overline{v})-t}\,)\,(\overline{v}-c(\overline{v}))>0$ when $p(t,\overline{v})=0$. When $p(t,\overline{v})=\overline{v}$, the term $(*)$ is equal to

$$g(\hat{t}) \equiv (1-e^{-r_{\rm b}\,\hat{t}}\,)\,\overline{v} - (1-e^{-r_{\rm s}\,\hat{t}}\,)\,c(\overline{v})$$

for $\hat{t} \equiv t^{v}(t,\overline{v})-t$. Note that $g(0)=0$ and $\lim_{\hat{t}\to\infty} g(\hat{t}) = \overline{v}-c(\overline{v}) > 0$. Simple analysis shows that $g'(\cdot)$ is single peaked, $\lim_{\hat{t}\to\infty} g'(\hat{t}) = 0$, and, by Assumption 2, we have $g'(0)>0$. Hence, the term $(*)$ is also positive when $p(t,\overline{v})=\overline{v}$. Henceforth, the term $(*)$ is positive for all $p(t,\overline{v}) \in [0,\overline{v}]$. □

(Proof of Proposition 3.2 continues.)

We now establish bounds on the seller's payoff, both from below and from above:

Lemma A.3. *For any t, $\overline{v}$, and $\overline{v}'$, with $\overline{v} > \overline{v}'$,*

$$\begin{aligned} &\tfrac{F(\overline{v})-F(\overline{v}')}{F(\overline{v})}\,\big(p(t,\overline{v}') - \mathbb{E}[c(\tilde{v})|\tilde{v}\in[\overline{v}',\overline{v}]]\big) + \tfrac{F(\overline{v}')}{F(\overline{v})}\,\Pi(t,\overline{v}') \\ &\qquad \le \Pi(t,\overline{v}) \le \tfrac{F(\overline{v})-F(\overline{v}')}{F(\overline{v})}\,\big(p(t,\overline{v}) - \mathbb{E}[c(\tilde{v})|\tilde{v}\in[\overline{v}',\overline{v}]]\big) + \tfrac{F(\overline{v}')}{F(\overline{v})}\,\Pi(t,\overline{v}')\;. \end{aligned} \tag{15}$$

Proof. The first inequality follows from the following observation. The seller has the option to "replicate" the continuation strategy she uses in state $(t,\overline{v}')$ when the state is, instead, $(t,\overline{v})$. By the same argument as in the proof of Lemma A.1, the seller obtains a payoff equal to the expression on left-hand side of the first inequality in (15) by doing so.

To prove the second inequality, recall Lemma A.2. Using the optimality of the seller's continuation strategy at $(t^{\overline{v}'},\overline{v}')$, we have that the right-hand side of expression (14) is no larger than

$$\tfrac{F(\overline{v})-F(\overline{v}')}{F(\overline{v})}\,\big(p(t,\overline{v}) - \mathbb{E}[c(\tilde{v})|\tilde{v}\in[\overline{v}',\overline{v}]]\big) + \tfrac{F(\overline{v}')}{F(\overline{v})}\,e^{-r_{\rm s}\,(t^{\overline{v}'}-t)}\,\Pi(t^{\overline{v}'},\overline{v}')\;, \tag{16}$$

and hence $\Pi(t,\overline{v})$ is smaller than expression (16). Note finally that, because the seller has the option of making unacceptable offers on $[t,t^{\overline{v}'})$, we have $e^{-r_{\rm s}\,(t^{\overline{v}'}-t)}\,\Pi(t^{\overline{v}'},\overline{v}') \le \Pi(t,\overline{v}')$, and hence the second inequality in expression (15) holds. □

(Proof of Proposition 3.2 continues.)

The following result establishes that $p(t,\cdot)$ is increasing and continuous:

Lemma A.4. *For all t, $p(t,\cdot)$ is increasing and continuous.*

Proof. **Proof that $p(t,\cdot)$ is increasing.** It follows directly from equation (15).

Proof that $p(t,\cdot)$ is continuous. The proof is similar to the proof of Proposition 3.1. We prove that $p(t,\cdot)$ is left-continuous (right-continuity is proven analogously). We do this by assuming, for the sake of contradiction, that $p(t,\cdot)$ is not left continuous at some $\overline{v}$; that is, there is a strictly increasing sequence $(\overline{v}_n)_n$ converging to $\overline{v}$ such that $p(t,\overline{v}_n) \to p_\infty \neq p(t,\overline{v})$. Since $p(t,\cdot)$ is increasing, it must be that $p_\infty < p(t,\overline{v})$. Let p^t be some history with $\overline{v}(p^t) = \overline{v}$, and we let $\overline{v}_{t'}$ denote $\overline{v}_{t'}(p^t)$. Also, for for each n, let p^t_n be a history with $\overline{v}(p^t_n) = \overline{v}_n$ (note they exist, see Footnote 7). Let finally t_n denote $t^{\overline{v}_n}(p^t)$, and note that $(t_n)_n$ (weakly) decreases toward t by Proposition 3.1.

We first prove that $t_n > t$ for all n. To see this, assume by contradiction that $t_{n-1} = t$ for some n, indicating that $\overline{v}_{t^+} \leq \overline{v}_{n-1}$. This implies that $\overline{v}_{t^+} < \overline{v}_n$. Consider the following continuation price path $\hat{p}_{(t,T]}$ defined by

$$\hat{p}_{t'} = \begin{cases} P_{t'}(p^t) & \text{if } t' \in (t, t+\varepsilon], \\ P_{t'}(p^t_{n+1}) & \text{if } t' \in (t+\varepsilon, T], \end{cases}$$

for some $\varepsilon > 0$. As ε shrinks towards 0, the seller's payoff from the previous continuation play at history p^t_n converges to

$$\frac{F(\overline{v}_n) - F(\overline{v}_{t^+}(p^t))}{F(\overline{v}_n)} \left(p(t,\overline{v}) - \mathbb{E}[c(\tilde{v}) | \tilde{v} \in [\overline{v}_{t^+}(p^t), \overline{v}_n]] \right) \\ + \frac{F(\overline{v}_{t^+}(p^t))}{F(\overline{v}_n)} \int_0^{\overline{v}_{t^+}(p^t)} e^{-r_s (t^v(p^t_n; P, a) - t)} \left(p^v(p^t_n; P, a) - c(v) \right) F(\mathrm{d}v) \ .$$

Given that $p(t,\overline{v}) > p(t,\overline{v}_n)$ and $F(\overline{v}_n) - F(\overline{v}_{t^+}(p^t)) > 0$, Lemma A.2 implies that the previous expression is strictly larger than $\Pi(t,\overline{v}_n)$. Since the seller has a profitable deviation, we reach a contradiction, and hence it must be that $t_n > t$ for all n.

Take some price $\hat{p} \in (p_\infty, p(t,\overline{v}))$. For each $t' > t$, let $\hat{v}_{t'} \equiv \overline{v}(p^t, P^{t'-}(p^t), \hat{p})$ be the upper valuation at time t' if the seller charges $\hat{p}$ at time t'. Consider a deviation of the seller after p^t, consisting in following the continuation path $P_{(t,t_n)}(p^t)$ on (t, t_n), then charging $\hat{p}$ at time t_n, and then continuing to follow the equilibrium strategy after t_n. Note that, since $p(t', \overline{v}_{t'})$ is continuous in t' on (t, t_n) (by the optimality of the buyer's strategy), we have that $p(t', \overline{v}_{t'}) > \hat{p}$ if n is big enough. There are two cases:

1. In the first case, there is no trade at time t_n when the seller offers $\hat{p}$ at time t_n; that is, $\hat{v}_{t_n} = \overline{v}_{t_n}$. In this case, the continuation play after t_n is unchanged. Nevertheless, this implies that an interval of buyer's types $[\underline{v}, \overline{v}_{t_n}]$, for some $\underline{v} < \overline{v}_{t_n}$, are willing to buy at prices larger than $\hat{p}$ at time t_n or later (recall that $p(t_n, \overline{v}_{t_n}) > \hat{p}$), a contradiction.

2. In the second case, trade occurs with positive probability at time t_n when the seller offers $\hat{p}$; that is, $\hat{v}_{t_n} < \overline{v}_{t_n}$. Now, there are two possibilities:

   (a) The first possibility is that $\hat{v}_{t_n} \leq \overline{v}_{t_n}(p^t_n)$, but this implies that

$$p(t_n, \hat{v}_{t_n}) = \hat{p} > p_\infty > p(t, \overline{v}_n) > p_{t_n}(t, \overline{v}_n) = p(t_n, \overline{v}_{t_n}(p^t_n)) \ ;$$

this contradicts that $p(t_n, \cdot)$ is increasing.

(b) The second possibility is that $\hat{v}_{t_n} > \overline{v}_{t_n}(p_n^t)$. Now, consider the following deviation of the seller on the continuation strategy at history p_n^t for n is large enough: the seller charges unacceptable prices in (t, t_n), then charges $\hat{p}$ at time t_n, and then offers $P_{t''}(t, \overline{v}_n)$ for all $t'' > t_n$. By the argument in the proof of Lemma A.1, using this strategy, the seller obtains the same payoff for all valuations in $[0, \hat{v}_{t_n}]$. Also, since the seller sells to the buyer when his valuation $[\hat{v}_{t_n}, \overline{v}_n]$ at time t_n at price $\hat{p}$, the deviation is profitable (note that when n is large, t_n is close to t, but $p_{t'}(t, \overline{v}_n) < p_\infty < \hat{p}$ for all $t' \in (t, t_n)$), which is again a contradiction.

□

(Proof of Proposition 3.2 continues.)

Define $\hat{\Pi}(t, \overline{v}) \equiv F(\overline{v})\, \Pi(t, \overline{v})$. From the first inequality in equation (A.4) and the continuity of $\pi(t, \cdot)$ (which follows from Lemma A.4 and the continuity of $c(\cdot)$) we have that

$$\liminf_{\overline{v}' \nearrow \overline{v}} \tfrac{\hat{\Pi}(t,\overline{v}) - \hat{\Pi}(t,\overline{v}')}{\overline{v} - \overline{v}'} \geq \pi(t, \overline{v})\, f(\overline{v}) \quad \text{and} \quad \liminf_{\overline{v} \searrow \overline{v}'} \tfrac{\hat{\Pi}(t,\overline{v}) - \hat{\Pi}(t,\overline{v}')}{\overline{v} - \overline{v}'} \geq \pi(t, \overline{v}')\, f(\overline{v}') \; . \tag{17}$$

Using the second inequality in equation (A.4) and, again, the continuity of $p(t, \cdot)$ we have that

$$\limsup_{\overline{v}' \nearrow \overline{v}} \tfrac{\hat{\Pi}(t,\overline{v}) - \hat{\Pi}(t,\overline{v}')}{\overline{v} - \overline{v}'} \leq \pi(t, \overline{v})\, f(\overline{v}) \quad \text{and} \quad \limsup_{\overline{v} \searrow \overline{v}'} \tfrac{\hat{\Pi}(\overline{v}) - \hat{\Pi}(\overline{v}')}{\overline{v} - \overline{v}'} \leq \pi(t, \overline{v}')\, f(\overline{v}') \; . \tag{18}$$

The four inequalities in expressions (17) and (18), together with the continuity of $p(t, \cdot)$ established in Lemma A.4, imply that $\Pi(t, \cdot)$ is differentiable, and the derivative is equal to

$$\tfrac{\mathrm{d}}{\mathrm{d}\overline{v}} \hat{\Pi}(t, \overline{v}) = \pi(t, \overline{v})\, f(\overline{v}) \; . \tag{19}$$

Integrating the previous expression gives equation (4). □

Proof of Proposition 3.3

Proof. Fix an equilibrium and history p^t. We write

$$\Pi(t, \overline{v}_t) = \tfrac{1}{F(\overline{v}_t)} \int_{[\overline{v}_t, \overline{v}_{t+\varepsilon})} e^{-r_{\mathrm{s}}\,(t^v - t)}\, \pi_{t^v}(t, \overline{v}_t)\, F(\mathrm{d}v) + \tfrac{F(\overline{v}_{t+\varepsilon})}{F(\overline{v}_t)}\, e^{-r_{\mathrm{s}}\,\varepsilon}\, \Pi(t+\varepsilon, \overline{v}_{t+\varepsilon}) \; ,$$

where $\pi_{t^v}(t, \overline{v}_t) \equiv p_{t^v}(t, \overline{v}_t) - c(v)$. Using equation (4), we have

$$0 = \underbrace{\int_{[\overline{v}_t, \overline{v}_{t+\varepsilon})} \left(\pi(t, v) - e^{-r_{\mathrm{s}}\,(t^v - t)}\, \pi_{t^v}(t, v)\right) F(\mathrm{d}v)}_{(*)} + F(\overline{v}_{t+\varepsilon}) \left(e^{-r_{\mathrm{s}}\,\varepsilon}\, \Pi(t+\varepsilon, \overline{v}_{t+\varepsilon}) - \Pi(t, \overline{v}_{t+\varepsilon})\right) .$$

Assume first $\overline{v}_t$ is right-differentiable at t (note that, given that $\overline{v}_{t'}$ is decreasing in time, it is differentiable for almost all times t'). In this case the term $(*)$ in the previous expression tends to 0 faster than ε as $\varepsilon \to 0$. Hence, in this case, the function $t \mapsto e^{-r_s t}\,\Pi(t,\overline{v})$ (for a fixed $\overline{v}$) is continuous and differentiable almost everywhere, with a derivative equal to 0 whenever it is differentiable. It then follows that it is derivative is equal to 0. This proves that $\Pi(t,\overline{v}) = e^{-(T-t)\,r_s}\,\Pi^*(\overline{v})$ as desired. □

Proof of Theorem 3.1

Proof. We first argue that the strategies of the seller and the buyer are mutual best responses. Indeed, equation (8) guarantees that the buyer's strategy is optimal. To see that the seller does not have the incentive to deviate, fix some state $(t,\overline{v})$ with $t < T$ and $\overline{v} > 0$ and a history p^t such that $\overline{v}(p^t) = \overline{v}$. Assume that the seller deviates at history p^t to some strategy $\hat{P}$. The continuation payoff from the deviation is

$$\int_0^{\overline{v}} e^{-r_s\,(t^v(p^t;\hat{P},a)-t)}\,(p^v(p^t;\hat{P},a) - c(v))\,F(\mathrm{d}v)\ .$$

By equation (7), the price paid by the v-buyer is at most

$$\left(1 - e^{-r_s\,(T-t^v(p^t;\hat{P},a))}\right) c(\overline{v}) + e^{-r_s\,(T-t^v(p^t;\hat{P},a))}\,p^*(\overline{v})\ .$$

Hence, the payoff the seller obtains from the deviation is no larger than

$$\int_0^{\overline{v}} e^{-r_s\,(T-t)}\,(p^*(\overline{v}) - c(\overline{v}))\,F(\mathrm{d}v)\ .$$

By Corollary 3.1 (which is proven independently of Theorem 3.1), the payoff the seller obtains from the deviation is no larger than $\Pi(t,\overline{v})$.

We proceed by showing that $\overline{v}_t$ solving the differential equation (9) (with the initial condition that $\overline{v}_t$ at time 0 is equal to the parameter $\overline{v}_0$) is decreasing. To verify this, we apply some algebra to equation (9) and obtain that

$$\dot{\overline{v}}_t = -\frac{r_b\,(\overline{v}_t - c(\overline{v}_t)) + e^{-r_s\,(T-t)}\,(r_s - r_b)\,(p^*(\overline{v}_t) - c(\overline{v}_t))}{(1 - e^{-r_s\,(T-t)})\,c'(\overline{v}_t) + e^{-r_s\,(T-t)}\,p^{*\prime}(\overline{v}_t)}\ .$$

From Assumption 1 we have that $p^*(\overline{v}_t)$ is increasing. Since $\overline{v}_t > c(\overline{v}_t)$ and $e^{-r_s\,(T-t)} \in (0,1]$, the right-hand side of the previous expression is not negative for all t and $\overline{v}_t$ if and only if

$$0 < r_b\,(\overline{v}_t - c(\overline{v}_t)) + (r_s - r_b)\,(p^*(\overline{v}_t) - c(\overline{v}_t))\ . \tag{20}$$

This is equivalent to Assumption 3.

The fact that Assumption 2 implies Assumption 3 follows from the fact that $p^*(\overline{v}_t) \in (0, \overline{v}_t)$ and that the right-hand side of the expression (20) is linear in $p^*(\overline{v}_t)$, equal to $r_{\mathrm{s}}\,(\overline{v}_t - c(\overline{v}_t)) > 0$ when $p^*(\overline{v}_t) = \overline{v}_t$, and equal to $r_{\mathrm{b}}\left(\overline{v}_t - \frac{r_{\mathrm{s}}}{r_{\mathrm{b}}}\, c(\overline{v}_t)\right)$ (which is positive by Assumption 2) when $p^*(\overline{v}_t) = 0$.

The uniqueness of the equilibrium follows from the arguments in Section 3.2, that clarify why the strategy profile described in statement of Theorem 3.1 is the unique candidate to be an equilibrium. □

Proof of Corollary 3.1

Proof. Using the envelope theorem, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}\overline{v}_0}\Pi^*(\overline{v}_0) = \tfrac{\mathrm{d}}{\mathrm{d}\overline{v}_0}\left(\int_{p^*(\overline{v}_0)}^{\overline{v}_0} (p^*(\overline{v}_0) - c(v))\, F(\mathrm{d}v)\right) = f(\overline{v}_0)\,(p^*(\overline{v}_0) - c(\overline{v}_0))\ .$$

It is then clear that the statement of the Corollary holds. □

Proof of Corollary 3.2

Proof. It follows from the proof of Theorem 3.1. □

A.3 Proofs of results in Section 4

Proof of Proposition 4.1

Proof. We first note that a change in r_{b} can be reformulated as a change in the unit used to measure time. To see this, fix some $\lambda > 1$. The model where the discount rate of the buyer is $\lambda\, r_{\mathrm{b}} > r_{\mathrm{b}}$, while all other parameters are the same, is referred to as the *λ-model*. The model where the discount rate of the seller is $r_{\mathrm{s}}^{\lambda} \equiv r_{\mathrm{s}}/\lambda < r_{\mathrm{s}}$ and the time horizon is $T^{\lambda} \equiv \lambda\, T > T$, while all other parameters are the same, is referred to as the *normalized λ-model*. Using $(p_t^{\lambda}, \overline{v}_t^{\lambda})$ to denote the equilibrium outcome of the λ-model, it is easy to see that the normalized λ-model has a unique equilibrium outcome, denoted $(p_t^{*\lambda}, \overline{v}_t^{*\lambda})$, and that this equilibrium outcome satisfies $(p_t^{*\lambda}, \overline{v}_t^{*\lambda}) = (p_{t/\lambda}^{\lambda}, \overline{v}_{t/\lambda}^{\lambda})$. As a result, both the seller and each valuation of the buyer obtain the same payoff in the λ-model and in the normalized λ-model.

Note that the product $r_{\mathrm{s}}^{\lambda}\, T^{\lambda}$ is independent of λ. From equations (7) and (8) we see that both $p_0^{*\lambda}$ and $\dot{p}_0^{*\lambda}$ are independent of λ as well. Furthermore, using equation (9), we have that

$$\tfrac{\mathrm{d}}{\mathrm{d}\lambda}\ddot{p}_0^{*\lambda} = -r_{\mathrm{b}}\,\tfrac{\mathrm{d}}{\mathrm{d}\lambda}\dot{\overline{v}}_0^{*\lambda} = -r_{\mathrm{b}}\,\frac{r_{\mathrm{s}}^{\lambda}\,(p^*(\overline{v}_0) - c(\overline{v}_0))}{\lambda^2\,((e^{r_{\mathrm{s}}^{\lambda}\,T^{\lambda}} - 1)\, c'(\overline{v}_0) - p^{*\prime}(\overline{v}_0))} < 0\,,$$

where we used that both c and p^* are increasing and also the assumption that adverse selection is not strong. Hence, since $\lambda > 1$, the price decreases faster around $t = 0$ in the normalized λ-model than in our base model. Finally, note the following. Assume that $p_t = p_t^{*\lambda}$ for some time $t \in (0, T)$. In this case, from equation (7), we have that

$$\left(1-e^{-r_{\rm s}\,(T-t)}\right)c(\overline{v}_t)+e^{-r_{\rm s}\,(T-t)}\,p^*(\overline{v}_t)=\left(1-e^{-r_{\rm s}^\lambda\,(T^\lambda-t)}\right)c(\overline{v}_t^{*\lambda})+e^{-r_{\rm s}^\lambda\,(T^\lambda-t)}\,p^*(\overline{v}_t^{*\lambda})\,. \tag{21}$$

Since $e^{-r_{\rm s}\,(T-t)} > e^{-r_{\rm s}^\lambda\,(T^\lambda-t)}$, both c and p^* are increasing, and since adverse selection is not strong, we have that $\overline{v}_t < \overline{v}_t^{*\lambda}$.[17] Hence, from equation (8), it follows that $\dot{p}_t > \dot{p}_t^{*\lambda}$. Nevertheless, standard ODE analysis implies that p_t and $p_t^{*\lambda}$ can only cross once, and such crossing time is $t=0$.[18] As a result, $p_t > p_t^{*\lambda}$ for all $t \in (0,T]$. Since the buyer's discount rate is the same in the normalized λ-model and in our base model (equal to $r_{\rm b}$), the buyer is better off in the normalized λ-model independently of his valuation, and therefore he is also better off in the λ-model than in our base model. In other words, the buyer is better off when he is more impatient. □

Proof of Proposition 4.2

Proof. The proof parallels the arguments in the proof of Proposition 4.1. Fix some $\lambda > 1$. We now define the *λ-model* as the model where the discount factor of the seller is $r_{\rm s}^\lambda \equiv r_{\rm s}/\lambda < r_{\rm s}$, while the rest of the parameters remain the same. We use $(p_t^\lambda, \overline{v}_t^\lambda)$ to denote the equilibrium outcome of the λ-model. Note that, because adverse selection is not strong, $p_0 < p_0^\lambda$. Assume, by contradiction, there is some $t \in (0,T)$ such that $p_t^\lambda = p_t$. Equation (21) now holds with $T^\lambda = T$. Since $e^{-r_{\rm s}\,(T-t)} < e^{-r_{\rm s}^\lambda\,(T-t)}$ (note that this inequality is reversed in the proof of Proposition 4.1), an argument analogous to the one in the proof of Proposition 4.1 (see footnote 17) implies that now $\overline{v}_t > \overline{v}_t^\lambda$, and hence $\dot{p}_t < \dot{p}_t^\lambda$. As in the proof of Proposition 4.1, this leads a contradiction, and so it $p_t < p_t^\lambda$ for all $t \in [0,T)$. Hence, all prices are higher in the λ-model than in our base model, and therefore the buyer is worse off when the seller is more patient. □

Proof of Proposition 4.3

Proof. The proof parallels the arguments in the proof of Propositions 4.1 and 4.2. Fix some $\lambda > 1$. We now define the *λ-model* as the model where the discount factor of the seller is $T^\lambda \equiv T\,\lambda > T$, while the rest of the parameters remain the same. We use $(p_t^\lambda, \overline{v}_t^\lambda)$ to denote the equilibrium outcome of the λ-model. Note that, because adverse selection is not strong, $p_0 > p_0^\lambda$. Assume, by contradiction, there

[17] Note that, since $e^{-r_{\rm s}\,(T-t)} > e^{-r_{\rm s}^\lambda\,(T^\lambda-t)}$, the weight on the static monopolistic price is larger on the left-hand side of equation (21). Since the static monopolistic price is larger than the cost (by no-strong adverse selection), and both are increasing functions, the threshold type of the right-hand side should be larger.

[18] Intuitively, if $p_t = p_t^{*\lambda}$ for some $t > 0$, we have that $\tilde{p}_t^{*\lambda}$ decreases faster, and hence $p_t^{*\lambda}$ crosses p_t "from above." Nevertheless, we also showed that p_t^λ is smaller than $p_t^{*\lambda}$ for low values of t. This implies that p_t and p_t^λ cross (at most) once, that is, at $t=0$.

is some $t \in (0, T)$ such that $p_t^\lambda = p_t$. Equation (21) now holds with $r_s^\lambda = r_s$. Since $e^{-r_s(T-t)} > e^{-r_s(T^\lambda - t)}$ (note that this inequality is now the same as in the proof of Proposition 4.1), an argument analogous to the one in the proof of Proposition 4.1 (see Footnote 17) implies that $\overline{v}_t < \overline{v}_t^\lambda$, and hence $\dot{p}_t > \dot{p}_t^\lambda$. As in the proof of Proposition 4.1, this leads a contradiction, and hence $p_t > p_t^\lambda$ for all $t \in [0, T)$. Hence, all prices are lower in the λ-model than in our base model, and therefore the buyer is better off when the time horizon is longer. □

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